Referees in 2021

The MAGAZINE expresses its appreciation to the following people for their help in refereeing during the past year (May 2020 to May 2021). It is with our utmost gratitude that we thank you for your volunteer work and service to the mathematics community. The quality content of *Mathematics Magazine* is sustained through the hard work of our editorial board members and referees. Thank You!

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LETTER FROM THE EDITOR

With this issue, I complete my second year as editor of Mathematics Magazine. In my first issue, February 2020, I wrote that I assumed this position with a mix of excitement and trepidation. Excitement because I was honored to take the helm of what I regard as one of the finest venues for expository mathematics in the world, but trepidation because the magazine receives hundreds of submissions every year, averaging more than one a day, and this presents a mammoth organizational challenge.

The past two years have seen great progress with regard to the smooth running of the magazine. I am pleased to report that processing times for new submissions are lower than they have been for some time, and authors now rarely have to wait more than two to three months to hear about their papers. I am incredibly grateful to my editorial board, and to an army of dedicated referees, for making this progress possible.

However, there is one problem I had not anticipated upon taking over the magazine: We receive too many good submissions. Even setting high editorial standards, there are so many talented authors and interesting ideas out there that we have amassed a substantial backlog of articles awaiting publication. As a result, some authors have had to wait a long time to see their accepted papers in print. We apologize to those authors, and we give our assurance that we are publishing items as quickly as possible. In 2022, our page count will go from 80 pages per issue to 84, and this will help to put a dent in the backlog. I suppose there are worse problems a magazine can have.

So, let us get down to our real business. We have the usual bumper crop of expository excellence for your enjoyment.

Our lead article is a remarkably personal look at cyclotomic polynomials, by Antonio Cafure. He relates his introduction to various concepts in abstract algebra, ultimately building up to a novel method for computing the constant terms of certain minimal polynomials. The article is fascinating not just for its mathematical content, but also for its strong emphasis on the human element in mathematics. I especially liked Cafure's opening. Like him, I also had an emotional reaction the first time I heard the term "cyclotomic polynomials." At the time I had no idea what they were, but the phrase sounded wonderfully hardcore.

Our co-lead article comes to us from Ethan Bolker, Samuel Feuer, and Catalin Zara. They take a serious look at a classic of recreational mathematics: balance weighing problems. They start with a problem accessible to fourth graders: finding the optimal sequence of weights needed to balance an object of unknown (but integral) weight. Their ensuing investigation, centered around a significant generalization of the problem, leads them into some pretty deep waters, but the authors' wonderfully clear writing makes this article well worth the effort.

The combinatorialists in the audience will enjoy the contribution from Craig Kaplan and Jeffrey Shallit. They find inspiration in a problem drawn from the area of discrete mathematics known as "combinatorics on words," eventually building up to an elegant construction of a frameless 2-coloring of the plane lattice. I will confess to being unfamiliar with this concept prior to reading their article, but their elegant presentation will make it accessible to all readers.

If you prefer your mathematics continuous, then have a look at Chuck Groetsch's article. His contribution blends history, physics, and calculus. He discusses Thomas

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Harriot's little-known sixteenth-century work on the problem of ballistic trajectories. Harriot got a lot of things right, especially relative to his predecessors, and his work deserves to be better known.

Michael Harrison's article revisits the classic "Grazing Goat" problem, introduced in the very first volume of *The American Mathematical Monthly* back in 1894. We imagine a goat grazing in a unit-circular field, tethered by a rope to a fence enclosing the field. We seek the length of rope enabling the goat to graze half the field. It seems like a tricky, but ultimately straightforward, exercise in integral calculus, but over the years a surprisingly dense literature has developed around it. Harrison presents a higher-dimensional version of the problem, the solution to which leads to some surprising properties of spheres.

It is back to combinatorics for our final article, a delightful note from Richard Ehrenborg. He starts with the old chestnut that any 2-coloring of the complete graph on six vertices contains a monochromatic triangle. It is less well-known that any such coloring must actually produce at least two monochromatic triangles, and this observation leads naturally to the question of the minimal number of monochromatic triangles in 2-colorings of higher order complete graphs. This problem was solved by Goodman in 1959, who produced the best possible lower bound. Ehrenborg produces a novel, and exceedingly clever, proof of this result.

We close out the issue with proofs without words from Roger Nelsen and Wei-Dong Jiang, problems and solutions, and reviews. A bumper crop to round out 2021.

Jason Rosenhouse, Editor

A Story About Cyclotomic Polynomials: The Minimal Polynomial of $2\cos(2\pi/n)$ and Its Constant Term

ANTONIO CAFURE
Universidad Nacional de General Sarmiento
Consejo Nacional de Investigaciones Científicas y Técnicas
Los Polvorines, Argentina
hombrealambre@gmail.com

As is apparent from the title, this article is a story about cyclotomic polynomials and how they are involved in the computation of the constant terms of the minimal polynomials of the numbers $2\cos(2\pi/n)$. I present here a new method for computing these numbers. The novelty of my approach relies on the fact that cyclotomic polynomials are particular cases of a more general family of polynomials: that of reciprocal polynomials. Taking this as a starting point, it is my intention also to share some of my personal experiences about learning, understanding, teaching and doing mathematics.

The beginning

I still remember with pleasure the first time I heard the words "cyclotomic polynomials," the sound of those words in my ears. I was an undergraduate student, and a friend of mine who was taking the field theory course (a course I had not yet taken) said them. I thought it seemed interesting and immediately asked him for details. He first told me to think a little about the word "cyclotomic": division of the circle. I said that one way of dividing the circle is by means of the nth roots of unity, and he nodded his head. "Well," he said, "consider now the nth primitive roots of unity $\omega_k = \cos(2k\pi/n) + i\sin(2k\pi/n)$, where k is coprime to n and $1 \le k < n$. The *nth cyclotomic polynomial* is the polynomial having them as roots:

$$\Phi_n(t) = \prod_{\substack{(k,n)=1 \\ 1\le k<n}} (t - \omega_k).$$

Immediately, I was able to work out some examples of cyclotomic polynomials, those corresponding to small values of n: Φ_1, Φ_2, Φ_3, Φ_4, Φ_5. These are shown in Table 1.

Moreover, it was clear that when n was a prime number, since all nth roots but 1 are primitive roots of unity, the cyclotomic polynomial was

$$\Phi_n(t) = t^{n-1} + t^{n-2} + \cdots + t + 1.$$

I also realized that the degree of Φ_n was equal to $\varphi(n)$, where φ is Euler's totient function. In my number theory course I had learned that $\varphi(n)$ counts the number of positive integers less than n which are coprime to n, and thus that the number of primitive nth roots of unity was equal to $\varphi(n)$.

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n	$\Phi_n(t)$
1	$t-1$
2	$t+1$
3	t^2+t+1
4	t^2+1
5	$t^4+t^3+t^2+t+1$
a prime	$t^{n-1}+t^{n-2}+\cdots+t^2+t+1$

TABLE 1: Some cyclotomic polynomials

I knew that for any divisor d of n, any dth root of unity is also an nth root. Then my friend showed me that by grouping together the factors $(t-\omega)$, with ω a primitive dth root of unity, the following identity holds:

$$t^n - 1 = \prod_{d|n} \prod_{\substack{\omega \text{ a } d\text{th} \\ \text{primitive root}}} (t-\omega) = \prod_{d|n} \Phi_d. \tag{1}$$

All the cyclotomic polynomials arising in our examples had integer coefficients. I asked him if this was always the case. Yes, he said, they are monic integer polynomials. He gave me a sketch of a proof by induction on n, starting from identity (1).

He commented on an amazing fact: cyclotomic polynomials are irreducible over $\mathbb{Q}$. However, the proof of this fact was not so easy to follow. Anyway, he mentioned that as a consequence of its irreducibility, Φ_n was the least degree monic polynomial in $\mathbb{Q}[t]$ having any primitive nth root of unity as a root. He had a precise definition for this: Φ_n is the *minimal polynomial* over $\mathbb{Q}$ of any primitive nth root of unity.

I could not help asking if the coefficients of cyclotomic polynomials were only $-1, 0, 1$. We had an infinite number of examples with such a behavior. He had presented the same question to his teaching assistant, who gave him the first n such that Φ_n has coefficients distinct from -1, 0 and 1: the cyclotomic polynomial Φ_{105} has -2 as the coefficient of the terms t^{41} and t^7. My friend also commented that he was studying a very interesting book (first published several years earlier) which for him was very well written, and, best of all, had many examples: *Abstract Algebra* ([**6**]), written by David Dummit and Richard Foote. Although there was no proof in the book, he read there that there were cyclotomic polynomials with arbitrarily large coefficients.

This talk was enough encouraging for me. I was eager to take the field theory course.

Taking the field theory course

The moment arrived and I finally took my field theory course. I learned about field extensions, about minimal polynomials, Galois groups, and finite fields. And of course, about cyclotomic polynomials and how to use them to compute *minimal polynomials* over $\mathbb{Q}$ of values of trigonometric functions: i.e. least degree monic polynomials with rational coefficients having these values as roots.

One of the assigned homework problems was to compute the minimal polynomial over $\mathbb{Q}$ of $2\cos(2\pi/7)$. For $k=1,\dots,6$, I considered $\omega_k = \cos(2k\pi/7) + i\sin(2k\pi/7)$, the primitive 7th roots of unity, the roots of Φ_7. Since each ω_k has modulus 1, its complex conjugate $\overline{\omega}_k$ (which equals ω_{7-k}) is equal to its inverse ω_k^{-1}. Thus, for $k = 1, 2, 3$, the following was true:

$$2\cos(2k\pi/7) = \omega_k + \overline{\omega_k} = \omega_k + \omega_{7-k} = \omega_k + \omega_k^{-1}.$$

In this way, I was able to relate ω_k to $2\cos(2k\pi/7)$. I learned a "trick" to link Φ_7 with the minimal polynomial of $2\cos(2\pi/7)$. The cyclotomic polynomial Φ_7 may be written as

$$\Phi_7(t) = t^3\Big(\big(t^3 + t^{-3}\big) + \big(t^2 + t^{-2}\big) + \big(t + t^{-1}\big) + 1\Big), \tag{2}$$

which in particular implies that

$$\big(\omega_k^3 + \omega_k^{-3}\big) + \big(\omega_k^2 + \omega_k^{-2}\big) + \big(\omega_k + \omega_k^{-1}\big) + 1 = 0.$$

If I was able to obtain a polynomial expression for $t^2 + t^{-2}$ and $t^3 + t^{-3}$ in terms of $x = t + t^{-1}$, then I would obtain a polynomial having $2\cos(2\pi/7)$ as a root, as well as $2\cos(4\pi/7)$ and $2\cos(6\pi/7)$. This was possible by writing

$$x^2 = (t + t^{-1})^2 = t^2 + t^{-2} + 2,$$
$$x^3 = (t + t^{-1})^3 = t^3 + t^{-3} + 3(t + t^{-1}).$$

Hence, I could deduce that $t^2 + t^{-2}$ and $t^3 + t^{-3}$ are polynomials in $\mathbb{Z}[x]$ of degree 2 and 3, respectively:

$$t^2 + t^{-2} = x^2 - 2 \quad \text{and} \quad t^3 + t^{-3} = x^3 - 3x.$$

I obtained an integer polynomial

$$C_7(x) = (x^3 - 3x) + (x^2 - 2) + x + 1 = x^3 + x^2 - 2x - 1,$$

whose roots are $\omega_k + \omega_k^{-1} = 2\cos(2k\pi/7)$, for $k = 1, 2, 3$. The final step was to show that C_7 was irreducible over $\mathbb{Q}$. One thing to observe was that equation (2) may be rephrased in terms of C_7 as

$$\Phi_7(t) = t^3 C_7(t + t^{-1}). \tag{3}$$

Thus, the irreducibility of Φ_7 over $\mathbb{Q}$ implied the same of C_7, and hence C_7 was the sought minimal polynomial.

With the same trick I could also compute many other minimal polynomials. For instance, C_5, C_9 and C_{11}, the minimal polynomial of $2\cos(2\pi/5)$, $2\cos(2\pi/9)$ and $2\cos(2\pi/11)$, respectively.

Having finished this course I was convinced that I wanted to go on studying these topics.

As a teaching assistant

Many years later, already a graduate student (studying polynomial systems over finite fields), and as the teaching assistant in a field theory course, I found *Polynomials*, the great book written by Edward Barbeau [2]. Reading this book, I realized that, except for $\Phi_1(x) = x - 1$, cyclotomic polynomials were examples of a more general family of polynomials: the set of reciprocal polynomials. Barbeau wrote: "A *reciprocal polynomial* has the form

$$f = at^n + bt^{n-1} + ct^{n-2} + \cdots + ct^2 + bt + a,$$

in which $a \neq 0$ and the coefficients are symmetric about the middle one." This means that the coefficient of t^j equals the coefficient of t^{n-j} for any j in the range $0 \leq j \leq n$. The trick I had learned in the context of cyclotomic polynomials was in fact a more general procedure for computing the roots of reciprocal polynomials, a topic which this book nicely introduced by means of exercises and explorations.

For instance,

$$f(t) = t^8 + 2t^7 - 12t^6 + 4t^5 - 11t^4 + 4t^3 - 12t^2 + 2t + 1$$

is a reciprocal polynomial. At first sight it is not evident how to compute its roots. Proceeding as in equation (2)

$$f(t) = t^4 \left((t^4 + t^{-4}) + 2(t^3 + t^{-3}) - 12(t^2 + t^{-2}) + 4(t + t^{-1}) - 11\right).$$

Setting again $x = t + t^{-1}$, the following identities are valid: $t^3 + t^{-3} = x^3 - 3x$, and $t^2 + t^{-2} = x^2 - 2$. It remains to compute $t^4 + t^{-4}$ as a polynomial in x. And this is not difficult since

$$t^4 + t^{-4} = (t + t^{-1})(t^3 + t^{-3}) - (t^2 + t^{-2}). \tag{4}$$

It turns out that

$$t^4 + t^{-4} = x(x^3 - 3x) - (x^2 - 2) = x^4 - 4x^2 + 2,$$

and thus, the polynomial

$$\begin{aligned} \mathsf{R}(f)(x) &= (x^4 - 4x^2 + 2) + 2(x^3 - 3x) - 12(x^2 - 2) + 4x - 11 \\ &= x^4 + 2x^3 - 16x^2 - 2x + 15 \end{aligned}$$

satisfies the identity

$$f(t) = t^4 \mathsf{R}(f)(t + t^{-1}).$$

The conclusion is that α is a root of f if and only if $\alpha + \alpha^{-1}$ is a root of $\mathsf{R}(f)$. In this case, it is easily seen that the roots of $\mathsf{R}(f)$ are ± 1, 3, -5. Therefore, to compute the roots of f we simply have to solve the four quadratic equations

$$t + t^{-1} = 1, \quad t + t^{-1} = -1, \quad t + t^{-1} = 3, \quad t + t^{-1} = -5.$$

This also yields the irreducible factorization of f over $\mathbb{Q}$:

$$f(t) = (t^2 - t + 1)(t^2 + t + 1)(t^2 - 3t + 1)(t^2 + 5t + 1).$$

The previous example illustrates the way in which reciprocal polynomials may be treated. Barbeau called this procedure the *reciprocal equation substitution*, and in its full generality it goes as follows: Any reciprocal polynomial $f \in \mathbb{Q}[t]$ of even degree, say $2n$, may be expressed like this:

$$f(t) = \sum_{k=1}^{n} a_k \left(t^{n+k} + t^{n-k}\right) + a_0 t^n = t^n \left(\sum_{k=1}^{n} a_k \left(t^k + t^{-k}\right) + a_0\right).$$

As before, such f will be related to a polynomial in $x = t + t^{-1}$ of degree n as long as each $t^k + t^{-k}$ could be expressed as a polynomial in x. Setting $t^0 + t^{-0} = 2$, thanks to the recursion

$$t^k + t^{-k} = \left(t + t^{-1}\right)\left(t^{k-1} + t^{1-k}\right) - \left(t^{k-2} + t^{2-k}\right),$$

k	$f_k(x)$
0	2
1	x
2	x^2-2
3	x^3-3x
4	x^4-4x+2
5	x^5-5x^3+5x

TABLE 2: The first polynomials $f_k(x)$

it is possible to prove by induction on k that t^k+t^{-k} is a monic polynomial $f_k \in \mathbb{Z}[x]$ of degree k, for every integer $k \geq 0$. Then the polynomials f_k satisfy the following recurrence:

$$f_k(x) = xf_{k-1}(x) - f_{k-2}(x), \quad f_0(x) = 2,\ f_1(x) = x.$$

This means that for any reciprocal polynomial f of degree $2n$, there exists a unique polynomial $\mathsf{R}(f) \in \mathbb{Q}[x]$ of degree n, which is computed as

$$\mathsf{R}(f)(x) = \sum_{k=1}^{n} a_k f_k(x) + a_0$$

and such that

$$f(t) = t^n \left(\sum_{k=1}^{n} a_k f_k(t+t^{-1}) + a_0 \right) = t^n \mathsf{R}(f)(t+t^{-1}). \tag{5}$$

The functional equation (5) is crucial for deducing many consequences of this procedure. First of all, α is a root of f if and only if $\alpha + \alpha^{-1}$ is a root of $\mathsf{R}(f)$. Moreover, if f turns out to be irreducible over $\mathbb{Q}$, then $\mathsf{R}(f)$ will also be irreducible.

Thanks to Barbeau's book I learned that there was a wider context in which what I had done in my field theory course was valid. Some time later I would learn even more.

Expanding the context

In 1956, the Mathematical Association of America first published *Irrational Numbers* by Ivan Niven. One of the interesting results which can be found there is the one stating that the numbers $2\cos(2\pi/n)$ are algebraic integers for every $n \in \mathbb{N}$ [9, Theorem 3.9]. A complex number ω is said to be an *algebraic integer* if there exists a monic integer polynomial having ω as a root. For instance, i is an algebraic integer (it is a root of x^2+1), while $\sqrt{2}/2$ is not (it is a root of $2x^2-1$).

One approach to proving the theorem may be finding a monic polynomial $C_n \in \mathbb{Z}[x]$ having $2\cos(2\pi/n)$ as a root. In some cases, the computation of C_n easily follows from simply computing $2\cos(2\pi/n)$, as shown in Table 3.

In others cases, the computation of C_n follows from what I did in my field theory course. In some of these cases we have a closed form for the value of $2\cos(2\pi/n)$ (as shown in Table 4, while in others we know nothing about it.

The proof Niven presents is due to Lehmer [8], and the relevant feature is that Lehmer's proof follows by considering cyclotomic polynomials as reciprocal polynomials. Certainly, Φ_n is reciprocal and has even degree for $n \geq 3$. Then by means of the

n	$2\cos(2\pi/n)$	$C_n(x)$
1	2	$x-2$
2	-2	$x+2$
3	-1	$x+1$
4	0	x
6	1	$x-1$
8	$\sqrt{2}$	x^2-2

TABLE 3: Easily computed polynomials C_n

n	$2\cos(2\pi/n)$	$C_n(x)$
5	$\frac{1}{2}(\sqrt{5}-1)$	x^2+x-1
7	?	x^3+x^2-2x-1
9	?	x^3-3x+1
10	$\frac{1}{2}(\sqrt{5}+1)$	x^2-x-1
11	?	$x^5+x^4-4x^3-3x^2+3x+1$

TABLE 4: Polynomials C_n computed by means of Φ_n and the reciprocal substitution

reciprocal substitution (terminology borrowed from Barbeau) each Φ_n is transformed into a monic polynomial $C_n = \mathsf{R}(\Phi_n) \in \mathbb{Z}[x]$ of degree $\varphi(n)/2$ having as roots the real numbers $2\cos(2k\pi/n) = \omega_k + \overline{\omega}_k$, with $(k, n) = 1$.

This already proves that the numbers $2\cos(2k\pi/n)$ are algebraic integers. Lehmer proves even more: the irreducibility of C_n following the same identity given by equation (5) (which encompasses equation (3)):

$$\Phi_n(t) = t^{\frac{\deg \Phi_n}{2}} C_n(t + t^{-1}) \quad \text{for } n \geq 3. \tag{6}$$

As with C_7, if C_n were reducible over $\mathbb{Q}$, then Φ_n would also be reducible. In this way, he also concludes that C_n is the minimal polynomial over $\mathbb{Q}$ of $2\cos(2k\pi/n)$.

In his book, Niven also made an interesting assertion with respect to the problem of computing and obtaining information about minimal polynomials of values of trigonometric functions. He stated that "the topic is a recurring one in the popular literature" [**9**, p. 41], (and consult the references therein). And as one may expect, Niven was right. These problems and its variants are (and probably will be) classical in number theory and elementary mathematics. Through the years there have been lots of articles dealing with this. It is difficult to provide an exhaustive overview of all of them, thus the following list is clearly arbitrary: Beslin and De Angelis [**3**], Cafure and Cesaratto [**4**] Gürtaş [**7**], Tangsuppathawat and Laohakosol [**11**], Watkins and Zeitlin [**12**], Wegner [**13**]. This also shows the importance of studying reciprocal polynomials as a way to proving properties of cyclotomic polynomials.

What this paper is about: Niven was right

I am finally in position to share with the readers a problem arising within the scope of cyclotomic polynomials and values of trigonometric functions, following Niven's vein. Since the numbers $2\cos(2\pi/n)$ are algebraic integers, it makes sense to pose the following question: which integers arise as constant terms of their minimal polynomials $C_n = \mathsf{R}(\Phi_n)$? In other words, which are the numbers $C_n(0) = \mathsf{R}(\Phi_n)(0)$?

My goal is to solve this problem from what I consider a simple point of view: that of considering cyclotomic polynomials as reciprocal polynomials. The method I present here, which I guess turns out to be another feature of the approach suggested by Lehmer and Niven, is also inspired by the article by myself and Cesaratto [4]. There, the usual reciprocal substitution $x = t + t^{-1}$ is interpreted as a one to one mapping from the set of reciprocal polynomials with rational coefficients of even degree onto $\mathbb{Q}[x]$, which is called the *reciprocal mapping*, and which is denoted by R. In fact, this article provides a general framework for dealing with problems involving reciprocal polynomials.

If f is any nonconstant reciprocal polynomial of even degree, then following equation (5) and the fact that the solutions of $t + t^{-1} = 0$ are $\pm i$, the constant term $\mathsf{R}(f)(0)$ may be computed as

$$\mathsf{R}(f)(0) = (-i)^{\frac{\deg f}{2}} f(i) \quad \text{or} \quad \mathsf{R}(f)(0) = (i)^{\frac{\deg f}{2}} f(-i). \tag{7}$$

This shows that the computation of $C_n(0)$ is an instance of equation (7), as equation (6) is a particular case of equation (5):

$$C_n(0) = (-i)^{\frac{\deg \Phi_n}{2}} \Phi_n(i) \quad \text{for } n \geq 3. \tag{8}$$

Well, according to equation (8) it seems that in order to obtain the integer sequence $C_n(0)$, I should compute the complex sequences $(-i)^{\frac{\deg \Phi_n}{2}}$ and $\Phi_n(i)$. First, in the next section, I recall some facts about reciprocal polynomials which will be very useful for the solution I propose. In the remainder of the article up to the Conclusions section, I also recover the usual *we*, which is the classical voice used in mathematics.

Reciprocal polynomials and the reciprocal mapping

Although we will only deal with reciprocal polynomials over the rationals, it is worth pointing out that the notion of "reciprocal polynomial" is valid over any field.

We begin by giving another definition of reciprocal polynomial, one which makes many things easier.

Definition (Reciprocal polynomial)**.** A polynomial $f \in \mathbb{Q}[t]$ is reciprocal if $f(t) = t^{\deg f} f(t^{-1})$.

As an immediate consequence of this definition, we observe that the set of reciprocal polynomials behaves well under multiplication. We can therefore quickly state some properties, which are exercises in Barbeau's book.

Lemma 1. *Let f and g be reciprocal polynomials. Then fg is reciprocal. If $f = gh$, then h is also reciprocal.*

Proof. Since f and g are reciprocal, we have that $f(t) = t^{\deg f} f(t^{-1})$ and $g(t) = t^{\deg g} g(t^{-1})$. It follows that

$$(fg)(t) = t^{\deg f} f(t^{-1}) t^{\deg g} g(t^{-1}) = t^{\deg f + \deg g} (fg)(t^{-1}),$$

showing that fg is reciprocal. Also, from

$$f(t) = t^{\deg f} (gh)(t^{-1}) = t^{\deg g} g(t^{-1}) t^{\deg h} h(t^{-1}) = g(t) t^{\deg h} h(t^{-1})$$

and unique factorization, we deduce that h is reciprocal. ∎

We have assumed that cyclotomic polynomials are reciprocal, but up to now we have given no proof of this. We may appeal to the definition since the inverse of any primitive nth root of unity is also a primitive nth root of unity. Anyway, we will provide a different proof of this fact, one of our own. The same type of reasoning (induction on n) which contributes to the proof that $\Phi_n \in \mathbb{Z}[t]$, for $n \in \mathbb{N}$, permits us to show that Φ_n is reciprocal for $n \geq 2$.

Proposition 1. *For $n \geq 2$, the cyclotomic polynomial Φ_n is reciprocal.*

Proof. If $n = 2$, then $\Phi_2(t) = t + 1$ is a reciprocal polynomial. Now, let n be greater than 2 and assume that Φ_d is reciprocal whenever $1 < d < n$. Thanks to equation (1) we have the following factorization:

$$t^{n-1} + t^{n-2} + \cdots + t + 1 = \prod_{\substack{d|n \\ d \neq 1}} \Phi_d(t).$$

If n is prime, then $\Phi_n(t) = t^{n-1} + t^{n-2} + \cdots + t + 1$ is reciprocal.

If n is not prime, then our inductive hypothesis and Lemma 1 imply that $\prod_{\substack{d|n \\ 1<d<n}} \Phi_d$ is reciprocal. Appealing again to Lemma 1, and using the factorization

$$t^{n-1} + t^{n-2} + \cdots + t + 1 = \Phi_n(t) \prod_{\substack{d|n \\ 1<d<n}} \Phi_d(t),$$

we conclude that Φ_n is a reciprocal polynomial. ■

The reciprocal equation substitution works for reciprocal polynomials of even degree. Thus, it seems appropriate to introduce a symbol to denote this set. We borrow the following notation from Cafure and Cesaratto [4]:

$$\mathcal{R} := \{f \in \mathbb{Q}[t] : \ f \text{ reciprocal and of even degree}\}.$$

It is clear that $\Phi_2(x) = x + 1$ does not belong to $\mathcal{R}$. What about Φ_n, for $n \geq 3$? Do they have even degree? Since $\deg \Phi_n = \varphi(n)$, this question may be rephrased as follows: Is $\varphi(n)$ even for $n \geq 3$? The answer is yes. If k is coprime with n, then so is $n - k$. Thus, the numbers coprime with n come in pairs.

We also borrow from Cafure and Cesaratto [4] the following definition.

Definition (Reciprocal mapping)**.** The reciprocal mapping R assigns to each polynomial $f \in \mathcal{R}$ of degree $2n$ the polynomial $\mathsf{R}(f) \in \mathbb{Q}[x]$ of degree n satisfying equation (5):

$$\mathsf{R} : \mathcal{R} \to \mathbb{Q}[x] \qquad f \mapsto \mathsf{R}(f).$$

We observe that R is the identity over $\mathbb{Q}$. Some useful properties of R are immediately derived from equation (5).

Proposition 2. R *has the following properties.*

1. R *is a bijective mapping.*
2. $\mathsf{R}(fg) = \mathsf{R}(f)\mathsf{R}(g)$ *for any $f, g \in \mathcal{R}$.*
3. *If $f \in \mathcal{R}$ is irreducible over $\mathbb{Q}$, then $\mathsf{R}(f)$ is irreducible over $\mathbb{Q}$.*

We have already pointed out that thanks to equation (8), the sequence of integer numbers $C_n(0)$ may be computed by means of the sequences of complex numbers $(-i)^{\frac{\deg \Phi_n}{2}}$ and $\Phi_n(i)$. Let us see to what extent this is absolutely necessary.

The case of prime powers

We concluded the previous section with the unanswered question about the computation of the sequences $(-i)^{\frac{\deg \Phi_n}{2}}$ and $\Phi_n(i)$.

It may be convenient to try some examples to get some ideas about what is going on. The prime case seems to be a good start since when p is prime, Φ_p is already known:

$$\Phi_p(t) = t^{p-1} + \cdots + t + 1 = \frac{t^p - 1}{t - 1}.$$

Moreover, Table 3 shows that $C_2(x) = x + 2$, and therefore it is enough to consider an odd prime p. Then $\Phi_p \in \mathcal{R}$ and the constant term $C_p(0)$ is obtained as

$$C_p(0) = (-i)^{\frac{p-1}{2}} \Phi_p(i).$$

Let us compute $\Phi_p(i)$ and $(-i)^{\frac{p-1}{2}}$ for an odd prime p. It is a good moment to recall that many facts involving odd prime numbers require considering them as split into two classes: the ones congruent to 1 modulo 4 and the ones congruent to 3 modulo 4. The computation of $\Phi_p(i)$ is our first example of this situation:

$$\Phi_p(i) = \frac{i^p - 1}{i - 1} = \begin{cases} 1 & \text{if } p \equiv 1 \mod 4 \\ i & \text{if } p \equiv 3 \mod 4. \end{cases} \tag{9}$$

Our second example is the computation of $(-i)^{\frac{p-1}{2}}$, which requires introducing $\lfloor \frac{p-1}{4} \rfloor$ to denote the integer part of $(p-1)/4$:

$$(-i)^{\frac{p-1}{2}} = \begin{cases} (-1)^{\lfloor \frac{p-1}{4} \rfloor} & \text{if } p \equiv 1 \mod 4 \\ (-1)^{\lfloor \frac{p-1}{4} \rfloor}(-i) & \text{if } p \equiv 3 \mod 4. \end{cases} \tag{10}$$

Combining equation (9) and equation (10) we conclude that the value of $C_p(0)$ is independent of the congruence class of the odd prime p. We have therefore solved the prime case:

$$C_p(0) = \begin{cases} 2 & \text{if } p = 2 \\ (-1)^{\lfloor \frac{p-1}{4} \rfloor} & \text{if } p \text{ is odd.} \end{cases} \tag{11}$$

The typical next stage is to study what happens with prime powers p^α, for $\alpha \geq 2$. It is natural to try relating Φ_p and Φ_{p^α}. As a first attempt, we could take identity (1) which provides a sort of connection between these polynomials:

$$t^{p^\alpha} - 1 = \prod_{k=0}^{\alpha} \Phi_{p^k}(t).$$

We readily see that we should start computing Φ_{p^2} since there is an inductive argument here. As we also have that

$$t^{p^2} - 1 = \Phi_1(t)\Phi_p(t)(t^{p(p-1)} + t^{p(p-2)} + \cdots + t^p + 1),$$

by unique factorization we conclude that

$$\Phi_{p^2}(t) = t^{p(p-1)} + t^{p(p-2)} + \cdots + t^p + 1 = \Phi_p(t^p).$$

Proceeding inductively, we can write

$$t^{p^\alpha} - 1 = \prod_{k=0}^{\alpha-1} \Phi_{p^k}(t) \left(\left(t^{p^{\alpha-1}}\right)^{p-1} + \left(t^{p^{\alpha-1}}\right)^{p-2} + \cdots + t^{p^{\alpha-1}} + 1 \right),$$

and again using unique factorization we conclude that

$$\Phi_{p^\alpha}(t) = \left(t^{p^{\alpha-1}}\right)^{p-1} + \left(t^{p^{\alpha-1}}\right)^{p-2} + \cdots + t^{p^{\alpha-1}} + 1.$$

Thus, for any prime p and any $\alpha \in \mathbb{N}$, we obtain the following identities:

$$\Phi_{p^\alpha}(t) = \Phi_{p^{\alpha-1}}(t^p) = \Phi_p(t^{p^{\alpha-1}}). \tag{12}$$

With this in hand, the constant term of C_{p^α} may be computed as

$$C_{p^\alpha}(0) = (-i)^{\frac{\deg \Phi_{p^\alpha}}{2}} \Phi_{p^\alpha}(i) = (-i)^{\frac{p^{\alpha-1}(p-1)}{2}} \Phi_p(i^{p^{\alpha-1}}).$$

We immediately deduce what happens when $p = 2$:

$$C_{2^\alpha}(0) = \begin{cases} 0 & \text{if } \alpha = 2 \\ -2 & \text{if } \alpha = 3 \\ 2 & \text{if } \alpha > 3. \end{cases} \tag{13}$$

However, when p is odd, although it is clear that $i^{p^{\alpha-1}}$ takes only the values i and $-i$, we would have to consider, as before, the congruence class of p. While the product of two numbers (in particular, two primes) of the form $4k + 1$ is again of the form $4k + 1$, the product of an even quantity of numbers of the form $4k + 3$ has the form $4k + 1$, and the product of an odd quantity has the form $4k + 3$. Hence we would have to consider not only p, but also the parity of α.

The interesting thing is that by exploiting the reciprocal nature of our polynomials, there will be no need of making these distinctions. This is the novelty of our approach. For any $f \in \mathcal{R}$ and any positive integer m, the composite polynomial $f(t^m)$ also belongs to $\mathcal{R}$. Before continuing, we introduce the following notation: $f^{(m)}(t) = f(t^m)$. With this notation, we write equation (12) as

$$\Phi_{p^\alpha}(t) = \Phi^{(p)}_{p^{\alpha-1}}(t) = \Phi_p^{(p^{\alpha-1})}(t).$$

Our next idea is to show how the computation of the constant term of $\mathsf{R}(f^{(m)})$ relies on computations involving only f.

Lemma 2. *Let $f \in \mathcal{R}$ be nonconstant and let m be a natural number. The constant term of $\mathsf{R}(f^{(m)})$ is computed as follows:*

$$\mathsf{R}(f^{(m)})(0) = \begin{cases} (-i)^{\frac{\deg f}{2}} f(i) & \text{if } m \equiv 1 \pmod 2 \\ f(1) & \text{if } m \equiv 0 \pmod 4 \\ (-1)^{\frac{\deg f}{2}} f(-1) & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

Proof. Substituting i^m for t in equation (5) and rephrasing in terms of our notation, we obtain that

$$\mathsf{R}(f)(i^m + i^{-m}) = (-i)^{\frac{m \deg f}{2}} f(i^m) = (-i)^{\frac{m \deg f}{2}} f^{(m)}(i). \tag{14}$$

At the same time, since $f^{(m)}$ belongs to $\mathcal{R}$, equation (7) implies that

$$\mathsf{R}(f^{(m)})(0) = (-i)^{\frac{m \deg f}{2}} f^{(m)}(i). \tag{15}$$

Combining equations (14) and (15), we deduce that

$$\mathsf{R}(f^{(m)})(0) = \mathsf{R}(f)(i^m + i^{-m}).$$

To finish the proof, we simply have to analyze the different cases arising for every $m \in \mathbb{N}$. When m is odd, the number $i^m + i^{-m}$ equals zero, and hence

$$\mathsf{R}(f^{(m)})(0) = \mathsf{R}(f)(0) = (-i)^{\frac{\deg f}{2}} f(i)$$

and the lemma holds.

For even m, the quantity $i^m + i^{-m}$ equals 2 or -2, according to whether m is a multiple of 4 or not, respectively. The lemma follows from equation (5) by substituting 1 for t or -1 for t, respectively. ■

The relevance of the previous lemma is that the computation of $\mathsf{R}(\Phi_p^{(m)})(0)$, for any $m \in \mathbb{N}$, relies only on the values of $\Phi_p(i)$, $\Phi_p(1)$ and $\Phi_p(-1)$. The first of these was obtained in equation (9), and the last two are easily computed.

Lemma 3. *If p is any prime, then $\Phi_p(1) = p$. If p is odd, then $\Phi_p(-1) = 1$ while $\Phi_2(-1) = 0$.*

For an odd prime number p, Lemma 2 states that

$$\mathsf{R}(\Phi_p^{(m)})(0) = \begin{cases} (-1)^{\lfloor (p-1)/4 \rfloor} & \text{if } m \equiv 1 \pmod 2 \\ p & \text{if } m \equiv 0 \pmod 4 \\ (-1)^{\frac{p-1}{2}} & \text{if } m \equiv 2 \pmod 4. \end{cases} \tag{16}$$

We are finally able to compute $C_{p^\alpha}(0)$ for any odd prime p and $\alpha \in \mathbb{N}$:

$$C_{p^\alpha}(0) = \mathsf{R}(\Phi_{p^\alpha})(0) = \mathsf{R}(\Phi_p^{(p^{\alpha-1})})(0) = (-1)^{\lfloor (p-1)/4 \rfloor} = C_p(0).$$

Recalling equations (11) and (13), we are in position to say that we have solved the problem for prime powers:

$$C_{p^\alpha}(0) = \begin{cases} 2 & \text{if } p = 2 \text{ and } \alpha \neq 2, 3 \\ 0 & \text{if } p = 2 \text{ and } \alpha = 2 \\ -2 & \text{if } p = 2 \text{ and } \alpha = 3 \\ (-1)^{\lfloor \frac{p-1}{4} \rfloor} & \text{if } p \text{ is odd.} \end{cases} \tag{17}$$

What is the next case to study? The one in which n has at least two different primes in its factorization.

Completing the computations

The remaining case to consider is that of np^α for a prime p coprime to n (which is greater than 1) and $\alpha \in \mathbb{N}$. In this section we focus on that, thereby completing our computations. To this end, we will introduce a classical result, a sort of complement to equation (1), which shows how to compute new cyclotomic polynomials from known ones. It also plays a very useful role in our approach and its proof is very simple.

Proposition 3. *Let $n \in \mathbb{N}$ and let p be a prime.*

1. *If p divides n, then $\Phi_{pn}(t) = \Phi_n(t^p)$.*
2. *If p does not divide n, then $\Phi_{pn}(t) = \dfrac{\Phi_n(t^p)}{\Phi_n(t)}$.*

Proof. If ω is any root of Φ_{pn}, then ω^p is a primitive nth root, and therefore ω is a root of $\Phi_n(t^p)$. This means that $\Phi_{pn}(t)$, which has degree $\varphi(pn)$, divides $\Phi_n(t^p)$, which has degree $p\varphi(n)$, in $\mathbb{Z}[t]$ (they are monic polynomials).

Now we introduce the issue of whether or not p divides n.

- If p divides n, then $\varphi(pn) = p\varphi(n)$. We conclude that the first assertion of our proposition holds.
- If p does not divide n, then $\varphi(pn) = (p-1)\varphi(n)$. Also, $\Phi_n(t)$ divides $\Phi_n(t^p)$ since λ^p is a root of Φ_n for any root λ of Φ_n. Since $\Phi_{pn}(t)$ and $\Phi_n(t)$ are coprime and $\varphi(pn) + \varphi(n) = p\varphi(n)$ we establish the validity of our second assertion.

■

First of all, we observe that the first item of Proposition 3 includes the particular case of prime powers given in equation (12). Second, suitably combining the identities provided by Proposition 3 we deduce that

$$\Phi_n(t^{p^{\alpha-1}})\ \Phi_{np^\alpha}(t) = \Phi_n(t^{p^\alpha}).$$

In our notation, this may be written as

$$\Phi_n^{(p^{\alpha-1})}(t)\ \Phi_{np^\alpha}(t) = \Phi_n^{(p^\alpha)}(t). \tag{18}$$

Now we introduce our point of view. If all the polynomials involved in equation (18) belong to $\mathcal{R}$, then that factorization is preserved by the reciprocal mapping R. Then the constant term $C_{np^\alpha}(0) = \mathsf{R}(\Phi_{np^\alpha})(0)$ is involved in the following equation:

$$\mathsf{R}(\Phi_n^{(p^{\alpha-1})})(0)\ C_{np^\alpha}(0) = \mathsf{R}(\Phi_n^{(p^\alpha)})(0).$$

To compute $\mathsf{R}(\Phi_n^{(p^{\alpha-1})})(0)$ and $\mathsf{R}(\Phi_n^{(p^\alpha)})(0)$, we can make use of Lemma 2, as long as $\Phi_n \in \mathcal{R}$. If this were the case, since we can also assume that p is odd, the last equation is equivalent to this other one:

$$\Phi_n(i)\ C_{np^\alpha}(0) = \Phi_n(i). \tag{19}$$

And of course it will turn out that $C_{np^\alpha}(0) = 1$ as long as $\Phi_n(i) \neq 0$. We have found two exceptions which prevent us from computing $C_{np^\alpha}(0)$ in full generality:

- Φ_n does not belong to $\mathcal{R}$ when $n = 2$.
- $\Phi_n(i) = 0$ if and only if $n = 4$.

Leaving aside these two exceptions, we are able to conclude that for $n \notin \{2, 4\}$, we have that $C_{np^\alpha}(0) = 1$. It remains only to study the exceptional cases $n = 2$ and $n = 4$. In each, we simply turn back to the definition of C_{2p^α} and C_{4p^α}. Thanks to Lemma 2 we have that

$$C_{2p^\alpha}(0) = \mathsf{R}(\Phi_{2p}^{(p^{\alpha-1})})(0) = (-i)^{\frac{p-1}{2}}\,\Phi_{2p}(i)$$

$$C_{4p^\alpha}(0) = \mathsf{R}(\Phi_{2p}^{(2p^{\alpha-1})})(0) = (-1)^{\frac{p-1}{2}}\Phi_{2p}(-1).$$

Proposition 3 and Lemma 3 together imply that

$$\Phi_{2p}(i) = \frac{\Phi_p(-1)}{\Phi_p(i)} = \frac{1}{\Phi_p(i)} \qquad \Phi_{2p}(-1) = \frac{\Phi_p(1)}{\Phi_p(-1)} = p.$$

Recalling equation (9) and equation (10) we conclude that

$$C_{2p^\alpha}(0) = \begin{cases} (-1)^{\lfloor (p-1)/4 \rfloor} & \text{if } p \equiv 1 \pmod 4 \\ (-1)^{\lfloor (p-1)/4 \rfloor + 1} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

We also easily deduce that

$$C_{4p^\alpha}(0) = (-1)^{\frac{p-1}{2}} p.$$

In this way, we have covered all possible cases. We gather all our partial results in the following theorem:

Theorem 1. *Let p be an odd prime and α be any positive integer. Then the constant term of C_n, the minimal polynomial over $\mathbb{Q}$ of $2\cos(2\pi/n)$, is:*

1. $C_1(0) = -2$.
2. $C_{2^\alpha}(0) = 2$ *if* $\alpha \neq 2, 3$, $C_4(0) = 0$ *and* $C_8(0) = -2$.
3. $C_{p^\alpha}(0) = (-1)^{\lfloor \frac{p-1}{4} \rfloor}$.
4. $C_{2p^\alpha}(0) = \begin{cases} (-1)^{\lfloor \frac{p-1}{4} \rfloor} & \textit{if } p \equiv 1 \pmod 4 \\ (-1)^{\lfloor \frac{p-1}{4} \rfloor + 1} & \textit{if } p \equiv 3 \pmod 4. \end{cases}$
5. $C_{4p^\alpha}(0) = (-1)^{\frac{p-1}{2}} p$.
6. $C_{np^\alpha}(0) = 1$ *for any natural n coprime with p and $n \notin \{2, 4\}$.*

Conclusions

We have finally arrived at the end of this article, and there are some words *I* would like to say.

In mathematical terms, the very novelty of the article is the solution I propose to the problem of computing the constant term $C_n(0)$. I have tried to provide as elementary a solution as possible, one that follows the ideas traced by Lehmer and Niven. I would feel very comfortable if I achieved that goal. In two recent papers, the authors compute $C_n(0)$ with arguments somewhat more involved than those presented here ([**1**,**5**]). The reciprocal polynomials approach really simplifies the computations when dealing with cyclotomic polynomials.

The integer sequence $C_n(0)$ is a subsequence of the https://oeis.org/A232624 sequence *Coefficient table for the minimal polynomials of* $2\cos(2\pi/n)$.

It was a surprise to learn about Proposition 3. I was not aware of it during my field theory course, and it took me some time to find it. Once I found it, I realized it is a kind of folklore result. For instance, in Prasolov [**10**, Theorem 3.3.5], one can find a proof of this result. Proposition 3 is the key to computing Φ_n at the fourth roots of unity for every n. At the same time, this belongs to a tradition of obtaining the values of cyclotomic polynomials at roots of unity.

I would like to mention something with respect to the irreducibility of cyclotomic polynomials. There is a nice article by Steven Weintraub [**14**] which gathers many classical proofs of this fact from such masters as Gauss, Eisenstein, Kronecker, Dedekind and others. Thanks to this article, I learned that the nowadays standard proof (see Dummit and Foote [**6**, Theorem 41, p. 554]) is a variant of Dedekind's proof.

I must say that the major challenge for me was how to narrate this story, how to share with readers my solution. During the process of writing I remembered the exact moment when I heard for the first time *cyclotomic polynomials*. That triggered the way in which I decided to write this article. To finish, I may assure you that the stories told in this article are almost quite true.

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Summary. I present a simple and novel way of computing the constant term of the minimal polynomial of the numbers $2\cos(2\pi/n)$. The method relies on computations with cyclotomic polynomials exploiting the fact that they are reciprocal polynomials. In addition, I tell a story about some of my experiences learning, teaching and doing mathematics.

ANTONIO CAFURE (MR Author ID: 780058) is a professor of mathematics at Universidad Nacional de General Sarmiento and a Researcher at CONICET. Apart from his mathematical activities, he also intends to improve his culinary skills: his family and friends enjoy that.

Balance Weighing: Variations on a Theme

ETHAN D. BOLKER
University of Massachusetts, Boston
Boston, MA 02125
ebolker@gmail.com

SAMUEL A. FEUER
Wesleyan University
Middletown, CT 06459
sfeuer45@gmail.com

CATALIN ZARA
University of Massachusetts, Boston
Boston, MA 02125
Catalin.Zara@umb.edu

A group of fourth graders at the Heath School in Brookline, MA, works to discover the optimal sequence of weights they would need to balance an object of unknown integral weight. They quickly discover the greedy algorithm for extending the sequence $(1, 2)$, realize that the next weight is one more than the sum of the ones found so far, and easily guess that it is double the last one found. Second graders can guess this too. Fourth graders can understand that knowing how to place the weights is equivalent to expanding integers in base 2.

Suppose you allow weights on either side of the balance. Then the optimal sequence begins $(1, 3)$ because $2 = 3 - 1$, so you can weigh a 2 pound object by putting it on the balance along with the 1 weight opposite the 3. The largest integer you can weigh is $4 = 1 + 3$. The next weight might be 6 instead of $5 = 4 + 1$ since $5 = 6 - 1$. With time and patience the fourth graders discover that 9 and then 27 are the right next weights. Since $1 + 3 + 9 + 27 = 40 = (3^4 - 1)/2$ they have solved the weight problem of Bachet de Méziriac:

> A merchant has a forty-pound measuring weight that broke into four pieces as the result of a fall. When the pieces were subsequently weighed, it was found that the weight of each piece was a whole number of pounds and that the four pieces could be used to weigh every integral weight between 1 and 40 pounds.
>
> What were the weights of the pieces? [**1**]

The problem reappears often. Figure 1 is from an early edition of Hugo Steinhaus's *Mathematical Snapshots* [**10**]. It was a Car Talk puzzler in 2011, reposted online in 2017 [**6**], [**7**]. It is a frequent visitor on StackExchange and other web question and answer sites [**3**], [**9**], and [**11**]. These references show that the following theorem is not new. This proof is implicit in the literature too.

Theorem 1. *When you may place weights on either side of the balance, you can weigh any integer uniquely using weights* $(3^k) = (1, 3, 9, \ldots)$.

Proof. To weigh n, write its base 3 representation. Then rewrite that representation using the digits $-1, 0, 1$ instead of the usual $0, 1, 2$ by repeating one of the following transformations as long as it is possible to do so:

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MSC: 11B75

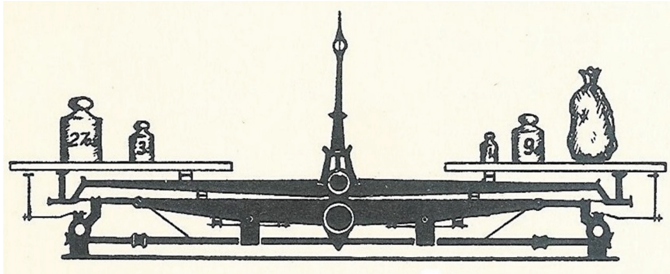

Figure 1 Steinhaus weighs in.

- Replace an instance of $x2y$ with $(x+1)(-1)y$.
- Replace an instance of $x3y$ with $(x+1)0y$.

Each of these replacements leaves the value of n invariant, the first because $2 \times 3^{k-1} = 3^k - 3^{k-1}$, the second because $3 \times 3^{k-1} = 3^k$.

The algorithm must terminate since the sum of the digits is strictly decreasing. The result tells you which weights to put on the balance along with the unknown n.

To convert back to the the usual base 3 representation

- Replace an instance of $x(-1)y$ with $(x-1)2y$.
- Replace an instance of $x(-2)y$ with $(x-1)1y$.

These changes do not change n, increase the digit sum, and end when the only digits are 0, 1 and 2. The existence of this inverse map shows that the balanced ternary representation, and hence the disposition of weights, is unique. ■

Using digits $0, \pm 1$ to represent integers in base 3 is called *balanced ternary* notation. It has a long history [2] , [5].

We posed a generalization to the students: What if you are allowed *at most one* weight along with the unknown object? We did not know the answer when we asked—intentionally, so kids could know that mathematicians regularly invented and attacked new questions that did not yet have answers. It is easy to see that the sequence of weights begins $(1, 3, 8)$; to weigh 5 you write $5 = 8 - 3$ rather than $5 = 9 - 3 - 1$ since the latter requires two negatives. The kids ran out of time before they could get any further, but we remained intrigued.

The start looks like every other Fibonacci number. We fantasized that the sequence might continue with 21. It does not: that would require two negative weights for $15 = 21 - 8 + 3 - 1$. After correcting several arithmetic errors, we found that the maximum next weight is 18. Watch the mystery weight sequence $X = (1, 3, 8, 18, \ldots)$ reappear later in this paper.

We hoped the Fibonacci pattern could be rescued by changing the problem—mathematicians do that all the time. Perhaps "using at most one negative weight" is the wrong hypothesis. So we tried "using at most half negative weights." Sadly, the weight sequence then starts $(1, 3, 8, 24, \ldots)$. The good news is that asking the more general questions led to some nice mathematics.

A framework for weighing problems

We shall start by restricting ourselves to weights on just one side of the balance.

Let $W = (w_0 < w_1 < w_2 < \cdots)$ be an increasing sequence of positive integers; we call such a sequence a *weight sequence*, and refer to the terms w_k as *weights*. We want to study weighing problems that allow for at most one weight of each kind. Since the theorems and proofs work just as well for a while with a bounded number $c \geqslant 1$ of each weight we shall state them in that generality.

Definition 1. *The weight sequence W is c-*complete *if every positive integer n is a sum of weights in W using each weight at most c times.*

Write $s_k = w_0 + w_1 + \cdots + w_{k-1}$ for the sum of the first k weights. Set $s_0 = 0$.

Theorem 2. *The weight sequence W is c-complete if and only if*

$$w_k \leqslant 1 + cs_k \tag{1}$$

for each $k \geqslant 0$.

Proof. The largest n you can weigh with the first k weights is cs_k. If the next weight is greater than $1 + cs_k$ then you cannot weigh $1 + cs_k$.

To prove the converse, suppose inequality (1) is true for all k.

For each nonnegative integer n let $r(n)$ be the base $c + 1$ representation of n: a string $d_k d_{k-1} \ldots d_0$ of digits between 0 and c. Then let $f(n)$ be the value of $r(n)$ in base W:

$$f(n) = d_k w_k + d_{k-1} w_{k-1} + \cdots + d_0 w_0.$$

Since $r((c+1)^k)$ is the string with a 1 followed by k 0's, $f((c+1)^k) = w_k$. In particular, that says f is unbounded.

If $r(n)$ ends in anything other than c, then $r(n+1)$ agrees with $r(n)$ except at the units digit, which increases by 1, so $f(n+1) = f(n) + 1$.

If $r(n)$ ends in a string of c's of length $j > 0$, then calculating $r(n+1)$ from $r(n)$ involves a carry to place $j + 1$, essentially replacing cs_j by w_j. Then inequality (1) implies

$$f(n+1) = f(n) + (w_j - cs_j) \leqslant f(n) + 1.$$

Thus f is an unbounded function with $f(1) = w_0 = 1$ and at each step f either increases by 1, remains unchanged, or decreases. Hence f is surjective. ■

The proof of Theorem 2 can be easily tweaked and generalized to prove slightly stronger results. Rewriting the argument in each of the following corollaries would obscure the bones of the argument, so we will leave some of the tweaks to the reader.

Corollary 1. *A weight sequence W is c-complete if and only if for every index $k \geqslant 0$, every positive integer $n \leqslant cs_{k+1}$ can be written as a sum of weights using only the weights $w_0, w_1, \ldots, w_k$ at most c times each.*

Proof. Note that

$$r((c+1)^k - 1) = ccc \ldots cc,$$

hence

$$f(((c+1)^k - 1)) = cs_k.$$

■

Corollary 2. *You need not allow the same maximum number of each kind of weight. The theorem has an obvious generalization when allowing c_k instances of weight k. The proof works with a mixed base representation using digits $0, 1, \ldots, c_k$ in column k.*

Corollary 3. *The weight sequence W is c-complete if $w_{k+1}/w_k \leqslant c+1$ for all k.*

Proof. Clearly $w_0 \leqslant 1 + cs_0$. We proceed by induction:

$$w_{k+1} \leqslant (c+1)w_k \leqslant cw_k + w_k \leqslant cw_k + 1 + cs_k = 1 + cs_{k+1}. \qquad \blacksquare$$

The condition $w_{k+1}/w_k \leqslant c+1$ is sufficient, but not necessary. For example, the sequence $(1, 2, 3, 7, 14, 28, \ldots)$ is 1-complete even though $7/3 > 2$.

Corollary 4. *As long as the weak inequality (1) is an equality (which may be true for all k) the weight sequence is the powers of $c+1$, and each n has a unique representation corresponding to its base $c+1$ expansion. Once there is a strict inequality there is occasional nonuniqueness.*

Using weights on both sides of the balance

In Theorem 1, we explored the relation between ternary and balanced ternary representations of integers: the digit sets $[0, 2] = \{0, 1, 2\}$ and $[-1, 1] = \{-1, 0, 1\}$ are essentially equivalent. That kind of equivalence persists when we allow multiple copies of each weight.

Lemma 1. *Suppose the weight sequence W represents n using $v(d)$ instances of coefficient digit d, for d in the digit set $[0, \ldots, 2c]$. Suppose further that $n < cs_k$. Then W represents $cs_k - n$ using digit $c - d$ in the digit set $[-c, \ldots, c]$ just $v(d)$ times. Conversely, that assertion remains true when you swap the roles of the two digit sets.*

Proof. Use the known representation of n to represent the complement:

$$\begin{array}{rcccccc} cs_k - n & = & cw_0 & +cw_1 & \cdots & +cw_k \\ & & -d_0w_0 & -d_1w_1 & \cdots & -d_kw_k \\ & = & (c-d_0)w_0 & +(c-d_1)w_1 & \cdots & +(c-d_k)w_k. \end{array}$$

The argument works the other way since the function $x \mapsto a - x$ is its own inverse; we use it with $a = cs_k$ for n and $a = c$ for the digit sets. $\blacksquare$

For example, $s_3 = 30$ for the mystery weight sequence $X = (1, 3, 8, 18, \ldots)$. The pair $(6, 24)$ exhibits these three sets of paired representations:

$$\begin{array}{rccc} 6 = & 8 - 3 + 1 = & 18 - 8 - 3 - 1 = & 2 \times 3 \\ 24 = & 18 + 2 \times 3 = & 2 \times 8 + 2 \times 3 + 2 \times 1 = & 18 + 8 - 3 + 1 \end{array}$$

This lemma provides an alternative algorithm for calculating the balanced ternary representation of n when you can use as many negative weights as you wish. For the powers of 3, $s_k = (3^k - 1)/2$. Find the smallest k such that $s_k > n$, find the ordinary base 3 representation of $s_k - n$ using $k+1$ digits and swap: interchange 0 and 1 and change 2 to -1.

Definition 2. *The weight sequence W is c-*balanced *if for every index $k \geqslant 0$, every positive integer $n \leqslant cs_{k+1}$ can be written as a sum*

$$n = d_kw_k + \cdots + d_0w_0$$

with digits d_i in $[-c, c]$.

We first proved the next theorem by modifying the proof of Theorem 2. Then we discovered Lemma 1, which makes for a much cleaner argument.

Theorem 3. *A weight sequence is c-balanced if and only if it is* $2c$*-complete.*

Proof. Let W be a weight sequence. Suppose $n > 0$. Choose k large enough so that $cs_k > n$. If W is $2c$-complete then $m = cs_k - n$ is a sum of weights using each at most $2c$ times. Then Lemma 1 implies n is a sum of weights using digits from $[-c, c]$. Conversely, if W is c-balanced, the lemma shows how to use the representation of m with those digits to represent n using $[0, 2c]$. ■

Restricting the number of negative weights

Now we are ready to return to open problems like the one we asked the fourth graders. Since we have just one weight of each kind to play with, we will restrict our discussion to the special case $c = 1$. Then we know that powers of 2 form the 1-complete weight sequence that leads to unique solutions while the sequence of powers of 3 does the same for the 2-complete, and hence 1-balanced, sequences.

We will continue to call weights that share the balance pan with the unknown "negative weights". Theorem 3 says we can isolate the object and use some "double weights" on the other pan instead, when that is more convenient

We want to allow the number of negative/double weights to depend on the number of weights we have available at any moment.

Definition 3. *A* bounding sequence *is a nondecreasing sequence of non-negative integers that increases by at most* 1 *at each step. Formally, it is a sequence* $\mu = (\mu_0, \mu_1, \mu_2 \cdots)$*, where* $\mu_0 = 0$ *and* $\mu_{k+1} - \mu_k \in \{0, 1\}$ *for all* $k > 0$*.*

A bounding sequence tells us that when using k weights we may put at most μ_k of them on the balance along with the unknown. The bounding sequence $\mu_{\min} = (0, 0, \ldots)$ allows no negative weights; the bounding sequence $\mu_{\max} = (0, 1, 2, \ldots)$ allows as many as you wish. The bounding sequence $(0, 1, 1, \ldots)$ allows just one negative weight.

Definition 4. *Let* μ *be a bounding sequence. A weight sequence* W *is* μ-balanced *if every* n *between* 1 *and* w_k *can be weighed using at most* μ_k *negative weights, that is, can be represented as*

$$n = d_k w_k + d_{k-1} w_{k-1} + \cdots + d_1 w_1 + d_0 w_0, \tag{2}$$

with digits in $[-1, 1]$ *at most* μ_k *of which are* -1*.*

Definition 5. *Let* μ *be a bounding sequence. A weight sequence* W *is* μ-complete *if every* n *between* 1 *and* $w_k - 1$ *can be weighed using at most* μ_k *double weights, that is, can be represented as*

$$n = b_{k-1} w_{k-1} + \cdots + b_1 w_1 + b_0 w_0,$$

with digits in $[0, 2]$ *at most* μ_k *of which are* 2*.*

Lemma 2. *A weight sequence* W *is* μ*-balanced if every* n *between* 1 *and* s_{k+1} *can be weighed using at most* μ_k *negative weights, that is, can be represented as*

$$n = d_k w_k + d_{k-1} w_{k-1} + \cdots + d_1 w_1 + d_0 w_0, \tag{3}$$

with digits in $[-1, 1]$ *at most* μ_k *of which are* -1*.*

Proof. The only difference between this lemma and the definition of μ-balanced is the change from $n \leq w_k - 1$ to $n \leqslant s_{k+1} - 1$.

One implication is clear, since $w_k < s_{k+1}$ for all k.

If $w_k \leqslant n < s_k$ then $n = w_k + m$ with $m \leqslant s_{k-1}$. Then by induction m is a sum of weights at most μ_k of which are negative; hence that's true for n as well. ■

Theorem 4. *Let μ be a bounding sequence. A weight sequence W is μ-balanced if and only if it is μ-complete.*

Proof. Suppose W is μ-balanced and $n < w_k$. Then $1 \leqslant n < s_{k+1}$, so $s_{k+1} - n$ can be represented using at most μ_k negative weights. Then Lemma 1 shows n can be represented using at most μ_k double weights.

Conversely, suppose W is μ-complete and $n \leqslant s_{k+1}$. If $n = s_j$ for some $j \leqslant k + 1$ then n is a sum of weights none of which is negative. If not, let j be the unique index such that $s_j < n < s_{j+1}$. Then apply Lemma 1 to the pair $(n, s_{j+1} - n)$. ■

Optimal weight sequences

Definition 6. *Let μ be a bounding sequence. A weight sequence W is μ-*optimal *if for every index $k \geqslant 0$, the value of w_k is the largest possible among μ-balanced/μ-complete weight sequences.*

Theorem 5. *For every μ there is a unique μ-optimal weight sequence W, constructed recursively as follows:*

- $w_0 = 1$*;*
- *Suppose $w_0, w_1, \ldots, w_{k-1}$ known. Then w_k is the smallest integer that can't be represented as a sum of smaller weights, allowing at most μ_k double weights.*

Then

$$2w_k \leqslant w_{k+1} \leqslant 3w_k$$

with equality on the left if and only if $\mu_k = 0$ and equality on the right if and only if $\mu_{k+1} = 1 + \mu_k$.

Proof. With only the first term of the sequence, all we can represent is w_0, so we must start with $w_0 = 1$. Suppose that we have determined optimal values for w_0, w_1, w_2, $\ldots$, w_k and want to find the optimal w_{k+1}.

Since W is μ-complete, all integers from 0 to $w_k - 1$ can be represented using weights up to w_{k-1} with the number of double weights bounded by μ. Then simply adding w_k to each representation weighs everything up to $2w_k - 1$ without using another double weight. Therefore $w_{k+1} \geqslant 2w_k$. That inequality can be strict only if we can weigh something larger. That is only possible if some earlier representation has a double weight we could make single so as to allow doubling the new next weight. That would require $\mu_k > 0$.

If $\mu_{k+1} = 1 + \mu_k$, then we may use another double weight. By adding $2w_k$ to the representation using weights up to w_{k-1} we can represent everything up to $3w_k - 1$, so $3w_k \leqslant w_{k+1}$.

Suppose we were able to represent $3w_k$ as well:

$$3w_k = d_k w_k + d_{k-1} w_{k-1} + \cdots + d_1 w_1 + d_0 w_0.$$

Since W is optimal, $w_k > s_k$, hence $3w_k > w_k + 2s_k$. Therefore, $d_k = 2$ and then

$$w_k = d_{k-1}w_{k-1} + \cdots + d_0 w_0,$$

using at most $\mu_{k+1} - 1 = \mu_k$ double weights. That contradicts the optimality of w_k.

Therefore, $3w_k$ is the smallest positive integer that cannot be represented by W using at most μ_k double weights, hence $w_{k+1} = 3w_k$. ■

The optimal weight sequence for $\mu_{\min}$ (no negative/double weights) is the powers of 2. The optimal sequence for $\mu_{\max}$ (as many negative/double weights as you wish) is the powers of 3. The problem we set the fourth graders was to find the μ-optimal weight sequence for $\mu = (0, 1, 1, 1, \ldots)$: one negative/double weight allowed. That is the mystery sequence $X = (1, 3, 8, 18, \ldots)$. We wrote a Python program to explore further. It told us the next two weights are 41 and 84. We found no pattern in X but did find X in OEIS at A066425 [8]:

$$W = (1, 3, 8, 18, 41, 84, 181, 364, 751, 1512, 3037, 6107, 12216, 24547, \ldots)$$

The next theorem shows that the OEIS definition for W is equivalent to ours for X.

Theorem 6. *For every integer $k \geqslant 0$, let w_k be the smallest positive integer M such that M minus any sum of distinct earlier terms is not already in the sequence. Then $w_k = x_k$, for all k.*

Proof. Use strong induction on k. The assertion is true for $w_0 = x_0 = 1$. Suppose it true up to $k - 1$.

Call the weights up to $k - 1$ "small".

Consider $m = w_k - 1$. The minimality of w_k implies there is some set S of small weights and a small weight $w_j = x_j$ such that $m - \Sigma S = w_j$ where ΣS is the sum of the weights in S. Then $m = w_j + \Sigma S$ represents m as a sum of small weights with at most one double weight. Therefore $m = w_k - 1 < x_k$ so $w_k \leqslant x_k$.

Consider $m = x_k - 1$. Then the optimality of x_k in X implies some set S of small weights represents m using at most one of them twice. If there is such a one, let it be w_j, else pick any element $w_j \in S$. Remove (one copy of) w_j from S (which may then be empty, but that is OK). Then $m = w_j + \Sigma S$. Then $m - \Sigma S = w_j$ so the minimality of w_k implies $m = x_k - 1 < w_k$ so $x_k \leqslant w_k$. ■

The OEIS entry asks for an efficient algorithm to calculate X. We have not found one. Our Python program is essentially brute force, speeded up some by the restrictions proved in Theorem 5.

Efficient algorithms are an unsolved problem. We do not have one to find any of the optimal sequences other than the powers of 2 and the powers of 3. Moreover, given an optimal sequence we do not have a good algorithm that tells us how to weigh things. With Lemma 1, we can switch back and forth between negative and double weights, but we do not know how to get started with either.

For the powers of 3 the greedy algorithm finds the ordinary base 3 representation: subtract as many as possible of the largest weight you can (1 or 2 of them), then represent the difference with smaller weights.

Since $w_{k+1} > 2w_k$ for every optimal weight sequence other than the powers of 2, the greedy algorithm will always call for 1 or 2 of the largest possible weight. But it may not succeed when 2 will fit. It fails to discover how X represents 38 since $38 = (2 \times 18) + (2 \times 1)$ uses two double weights. The correct representation is

$$38 = 18 + 2 \times 8 + 3 + 1.$$

We hoped that the greedy algorithm would work at least for $w_k - 1$, the last n you can represent without needing a new weight. But no: it fails for X at

$$w_{11} - 1 = 6106 = 3037 + 2 \times 1512 + 41 + 3 + 1$$

even though $2 \times 3037 < 6107$.

Conjecture 1. *For every optimal weight sequence other than the powers of 2 and powers of 3, using the greedy algorithm to find representations will always fail somewhere. The first failure will be for an n for which the largest possible weight fits twice.*

We can modify the greedy algorithm by adding a little backtracking. If at some point subtracting a double weight leads to failure, try a single instead. This is essentially the well-known (in computer science) recursive solution to the knapsack problem [4]. Starting with the largest weight is just a way to organize the brute force search through all the possibilities (which must succeed) in hopes that reducing the problem recursively by subtracting large weights will find a solution as quickly as possible.

The tree of optimal weight sequences

When we computed optimal weight sequences for several bounding sequences μ, we discovered that although we could not easily understand any particular sequence, studied together the set of weight sequences displays surprising structure.

Given a bounding sequence μ, the sequence of differences $\mu_{k+1} - \mu_k \in \{0, 1\}$ determines μ, and any infinite sequence of 0s and 1s determines a bounding sequence. We will use the sequence of differences to identify bounding sequences and hence optimal weight sequences.

Definition 7. *For the finite bit string Δ of length k, let $w(\Delta)$ be the weight w_k for any of the optimal weight sequences whose bounding sequence differences begin with Δ. Theorem 5 guarantees that $w(\Delta)$ is independent of the choice of bounding sequence. Set $W(\text{empty string}) = 1$.*

Then for example $w(\mathtt{0}) = 2$, $w(\mathtt{1}) = 3$, and $w(\mathtt{1000}) = 41$.

It is natural to display data determined by finite bit sequences at the nodes of a binary tree: `0` moves to the left child, `1` to the right.

Figure 2 shows part of the tree that in its infinite entirety would display all the optimal weight sequences. Each path from the root corresponds to a sequence of `0`s and `1`s, hence to an infinite bit string Δ, a μ and the μ-optimal weight sequence recorded in the nodes along the path. The mystery weight sequence X is the leftmost branch in the right half of the tree, corresponding to $\Delta = \mathtt{1000}\ldots$.

Visible patterns in the tree lead at once to the following.

Theorem 7. *Let Δ be a finite bit string. Write $\mathtt{b}\Delta$ and $\Delta\mathtt{b}$ for pre- and post- concatenation with bit `b`. Then*

1. *The left half of the tree below $w(\mathtt{0}) = 2$ is double the whole tree: $w(\mathtt{0}\Delta) = 2w(\Delta)$.*
2. *Following a right branch triples the weight: $w(\Delta\mathtt{1}) = 3w(\Delta)$.*
3. *Following a left branch at least doubles but less than triples the weight:*

$$2w(\Delta) \leqslant w(\Delta\mathtt{0}) < 3w(\Delta).$$

Proof. Suppose Δ is of length k, so determines the weights up to $w_k = w(\Delta)$ in one of the optimal weight sequences W that start on the path determined by Δ.

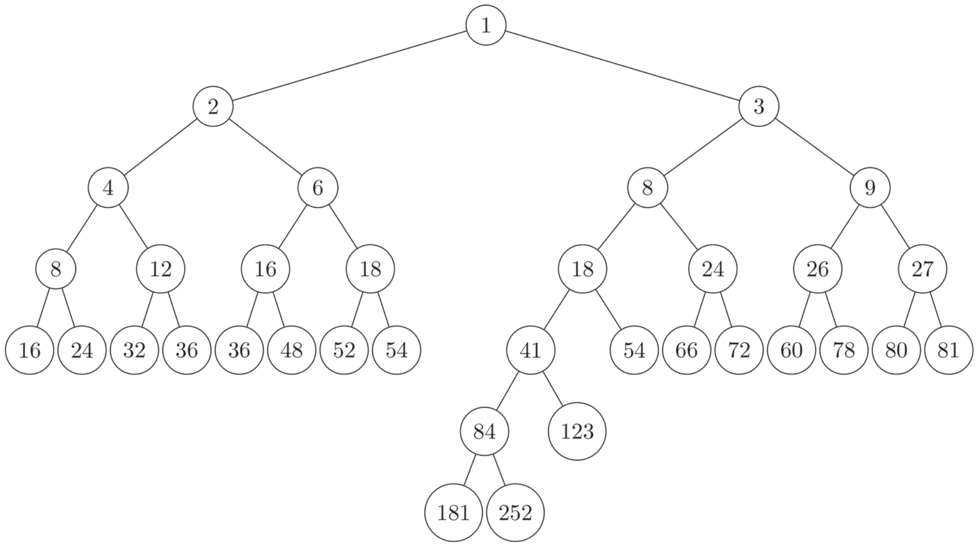

Figure 2 The tree of optimal weight sequences.

To prove the first assertion, imagine constructing the tree starting with $w_0 = 2$ rather than 1. It is clear that you will then be able to weigh all even integers optimally up to $2w(\Delta) - 2$ while respecting the number of double or negative weights specified by Δ. If you then view $\mathtt{0}\Delta$ as a path in the full tree you are allowed one more weight, but no more double weights. Use that extra weight to add the 1 at the root to the even numbers you already know how to represent in order to represent all the integers up to any $2w(\Delta) - 1$.

The second and third assertions follow immediately from Theorem 5 . ∎

It is natural to analyze data in a tree by following paths from the root. It is less natural to look for patterns between paths, but there are many in this tree. The first we noticed concerns the differences between the weights of siblings. (We will often shorten expressions like "difference between weights of siblings" by the slightly less precise "difference between siblings.")

Theorem 8 (Siblings)**.** *When you follow a right branch in the tree the difference between siblings is constant. Formally, for each finite bit string* Δ

$$w(\Delta\mathtt{11}) - w(\Delta\mathtt{10}) = w(\Delta\mathtt{1}) - w(\Delta\mathtt{0}). \tag{4}$$

Proof. Theorem 7 (2) says $w(\Delta\mathtt{11}) = 3w(\Delta\mathtt{1})$ so Equation (4) is equivalent to

$$w(\Delta\mathtt{10}) = 2w(\Delta\mathtt{1}) + w(\Delta\mathtt{0}).$$

Let $w_0, w_1, \ldots, w_k, w_{k+1}$ be the weight list corresponding to $\Delta\mathtt{1}$ and let $m = \mu(\Delta)$ be the sum of the entries of Δ; then $w_k = w(\Delta)$ is the smallest integer that can not be weighed with $w_0, \ldots, w_{k-1}$ with at most m double weights.

Let

$$N = 2w(\Delta\mathtt{1}) + w(\Delta\mathtt{0})$$

and suppose N can be represented as

$$N = d_0 w_0 + \cdots + d_k w_k + d_{k+1} w_{k+1} \tag{5}$$

with at most $\mu(\Delta 10) = m + 1$ coefficients equal to 2. Since

$$w(\Delta 0) > w_0 + \cdots + w_k$$

and

$$w(\Delta 1) > w_0 + \cdots + w_k,$$

it follows that

$$N = w(\Delta 0) + w(\Delta 1) + w(\Delta 1) > 2(w_0 + \cdots + w_k) + w_{k+1}.$$

That implies $d_{k+1} = 2$ and therefore

$$w(\Delta 0) = N - 2w(\Delta 1) = d_0 w_0 + \cdots + d_k w_k,$$

with at most $m = \mu(\Delta 0)$ coefficients equal to 2, contradicting the definition of $w(\Delta 0)$. Hence,

$$N = 2w(\Delta 1) + w(\Delta 0)$$

does not have a representation as in Equation (5).

Next, we show that every positive integer $n < N$ has such a representation. We consider two cases:

Case 1: $n \leqslant 2w(\Delta 1)$. Then $n < w(\Delta 10)$, hence n can be represented as

$$n = d_0 w_0 + \cdots + d_k w_k + d_{k+1} w_{k+1}$$

with at most $m + 1 = \mu(\Delta 10)$ coefficients equal to 2.

Case 2: $2w(\Delta 1) < n < 2w(\Delta 1) + w(\Delta 0)$. Then $n - 2w(\Delta 1) < w(\Delta 0)$, hence it can be represented as

$$n - 2w_{k+1} = d_0 w_0 + \cdots + d_k w_k,$$

with at most $\mu(\Delta 0) = \mu(\Delta) = m$ coefficients equal to 2. Therefore,

$$n = d_0 w_0 + \cdots + d_k w_k + 2w_{k+1},$$

with at most $1 + \mu(\Delta 0) = \mu(\Delta 10)$ coefficients equal to 2.

Therefore,

$$N = 2w(\Delta 1) + w(\Delta 0)$$

is the smallest positive integer that does not have a representation (5), hence $N = w(\Delta 10)$. ■

First cousins are nodes with a different parent but a common grandparent. For each node, the right child of the left child and the left child of the right child are neighboring first cousins.

Corollary 5 (Cousins)**.** *The difference between neighboring first cousins is twice the difference between their sibling parents. Formally,*

$$w(\Delta 10) - w(\Delta 01) = 2(w(\Delta 1) - w(\Delta 0).$$

Proof. Let

$$a = w(\Delta 1) - w(\Delta 0).$$

Then

$$\begin{aligned} w(\Delta 10) - w(\Delta 01) &= w(\Delta 10) - 3w(\Delta 0) \\ &= w(\Delta 11) - a - 3w(\Delta 0) \\ &= 3w(\Delta 1) - a - 3w(\Delta 0) \\ &= 3a - a = 2a. \end{aligned}$$

■

With the results in this section, we can begin to fill in the tree using only elementary arithmetic and the weights in the mystery sequence.

- At the root, we know the value is 1.
- Theorem 7 fills the second row with $w(0) = 2$ and $w(1) = 3$. That tells us the left half of the third row too: $w(00) = 4$ and $w(01) = 6$. The right half of the third row is $w(11) = 9$ (Theorem 7) and $w(10) = 8$ (Siblings).
- The right half of the fourth row starts with $w(100) = 18$ from the mystery sequence. Theorem 7 tells us $w(101) = 24$ and $w(111) = 27$. Then the sibling theorem or the cousin corollary finishes with $w(110) = 26$.
- Similar reasoning fills the fourth row except for $w(1000) = 41$, from the sequence X, and $w(1100) = 60$, which seems to require an actual search for the optimal weight.

We hoped that the mystery sequence X might tell us $w(1100) = 60$, in fact everything, if only we we could generalize Corollary 5 to more distant cousins.

The rest of this paper shows how that hope played out.

Kissing cousins

In a binary tree, a pair of nodes in a row are kth cousins for their nearest common ancestor node N if they live $k + 1$ rows below N. 0th cousins are siblings. There are 2^{2k} pairs of of kth cousins with common ancestor N since you form such a pair by choosing one cousin from the left half of the tree below N and the other from the right half.

Among those pairs we single out 2^{k-1} pairs of *kissing cousins*. If you number the nodes in row $k + 1$ below N as $1, 2, \ldots, 2^k$ in the left and right halves then the kissing kth cousins are the pairs

$$(2, 1), (4, 2), (6, 3), \ldots, (2^k, 2^{k-1}) .$$

Figure 3 marks the pairs of kissing cousins below the root with solid lines and those below node 1 (weight 3) with dashed lines.

This figure suggests the conjecture that kissing kth cousin differences propagate when you travel down to the right just as first cousin differences do. For example,

$$w(100) - w(001) = 18 - 12 = 60 - 54 = w(1100) - w(1001).$$

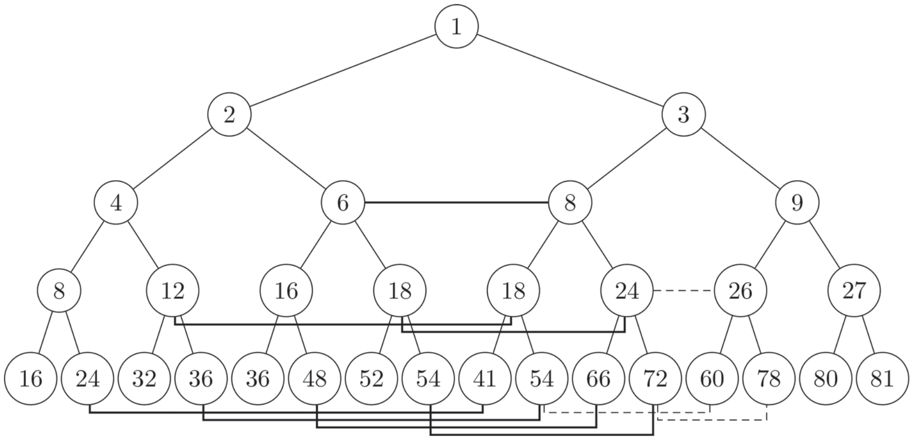

Figure 3 Kissing cousins.

We have much more data supporting that conjecture If it is true we can use it to extend what's possible with just simple arithmetic and the weights in X. In particular, we can fill in the missing fourth row value $w(\mathtt{1100}) = 60$.

Unfortunately we will still be stuck trying to find $w(\mathtt{10100})$ in the fifth row. The two terminal moves left stymie us and there is no pair of kissing cousins to help out.

Since the conjecture does not tell us everything, we content ourselves with stating it formally and drawing some consequences, and leave the proof to the reader.

Definition 8. *For each bit string Δ we define the kissing cousin pairs below node Δ recursively.*

The pair $(\Delta\mathtt{01}, \Delta\mathtt{10})$ *are* kissing first cousins.

For each pair $(\Delta\mathtt{0X1}, \Delta\mathtt{1Y})$ *of kissing kth cousins there are two pairs of kissing* $(k+1)$*st cousins:*

$$\textit{left: } (\Delta\mathtt{0X01}, \Delta\mathtt{1Y0})$$

and

$$\textit{right: } (\Delta\mathtt{0X11}, \Delta\mathtt{1Y1}).$$

Note that Y *is one bit longer than* X*. For the kissing first cousins,* X *is empty and* $\mathtt{Y} = \mathtt{0}$*. For kth cousins,* $|Y| = k$*.*

Conjecture 2 (Kissing cousins)**.** *The difference between kissing cousins is invariant when you move their common great-parent down a right branch. Formally, for each bit string Δ, for each pair of kissing cousins*

$$w(\Delta\mathtt{11Y0}) - w(\Delta\mathtt{10X01}) = w(\Delta\mathtt{1Y0}) - w(\Delta\mathtt{0X01}). \tag{6}$$

Definition 8 shows that for each Δ the pairs of kissing cousins naturally arrange themselves in a binary tree with root $(\Delta\mathtt{01}, \Delta\mathtt{10})$. In that tree we define the value at each node to be the difference between the weights of the pair of cousins at that node. When we wrote out the first few levels of that tree for several fixed ancestors Δ we discovered that it too seemed to satisfy Conjecture 2. That is not an accident.

Theorem 9. *Let T be a binary tree in which following a right branch triples the value at a node and the kissing cousin conjecture holds. Then the same is true for the trees $K(\Delta)$ of differences between kissing cousin pairs below Δ in T.*

Proof. Write $t(N)$ for the value at node N in T. Following a right branch in $K(\Delta)$ triples because for each X and Y defining a kissing cousin pair

$$t(\Delta\mathtt{1Y1}) - t(\Delta\mathtt{0X11}) = 3t(\Delta\mathtt{1Y}) - 3t(\Delta\mathtt{0X1}) = 3(t(\Delta\mathtt{1Y}) - t(\Delta\mathtt{0X1})).$$

If $(\Delta\mathtt{A}, \Delta\mathtt{B})$ and $(\Delta\mathtt{C}, \Delta\mathtt{D})$ are kissing cousins in $K(T)$ then

$$\begin{aligned} t(\Delta\mathtt{1B}) - t(\Delta\mathtt{1A}) &- (t(\Delta\mathtt{1D}) - t(\Delta\mathtt{1C})) \\ &= t(\Delta\mathtt{1C}) - t(\Delta\mathtt{1A}) - (t(\Delta\mathtt{1D}) - t(\Delta\mathtt{1B})) \\ &= t(\Delta\mathtt{C}) - t(\Delta\mathtt{A}) - (t(\Delta\mathtt{D}) - t(\Delta\mathtt{B})). \end{aligned}$$

The last equality is true because the terms on the right are kissing cousin differences in T, equal because they come from following the right branch appending `1` to Δ. ■

The tree of kissing cousin differences depends on the chosen common ancestor Δ, which you can think of as the head of a family, so taken together those trees are a forest of (overlapping) family trees. Examining the tree headed by $w(\mathtt{10}) = 8$ shows that we can't conjecture some further resemblances with the weight tree. In particular, there are places when the left kissing cousin difference is greater than three times the parent cousin difference. There is much more to be explored in these forests.

What next?

We started with an unanswered fourth grade balance weighing question: what happens when you allow just one weight on the pan with the unknown?

Explorations at that elementary level revealed the power of positional notation as a tool and a metaphor.

We showed how answers to generalizations of that question live in an interesting, richly structured tree of weights.

There is more nice mathematics waiting to be discovered (or invented, depending on your philosophy of mathematics):

- Find good algorithms for calculating optimal weight sequences, or prove that these calculations are intrinsically difficult, essentially calling for brute force search. In particular, understand the mystery sequence $X = (1, 3, 8, 18, 41, \dots)$ that answers the fourth grade open question.
- Given an optimal weight sequence, find a good algorithm for how to weigh with it, or prove that these calculations are intrinsically difficult.
- Prove the kissing cousin conjecture.
- Think about which weight sequences are optimal for some bounding sequence.
- Generalize to allow multiple (but bounded in number) weights of each size.

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Summary. We explore weighing problems when you may use at most one of each of an increasing sequence of weights, but may put some on the balance along with the unknown object. Our solutions depend on analyzing arithmetic when you expand integers in the mixed base defined by the weights. We begin with elementary school exercises and end with conjectures.

ETHAN D. BOLKER (MR Author ID: 38955) is Professor of Mathematics, Emeritus, at the University of Massachusetts, Boston. His favorite research topics come from elementary mathematics and ideas he first encountered in Hugo Steinhaus' *Mathematical Snapshots*.

SAMUEL A. FEUER is a freshman at Wesleyan University. When he was six, he discovered that he loved mathematics and music. They are still his greatest passions today.

CATALIN ZARA (MR Author ID: 646469) is Professor of Mathematics at the University of Massachusetts, Boston. His ongoing passion for solving math problems traces back to early years of school in Romania. He credits this passion to his parents for starting it and his mentors for nurturing it.

A Frameless 2-Coloring of the Plane Lattice

CRAIG S. KAPLAN
University of Waterloo
Waterloo, ON, Canada
csk@uwaterloo.ca

JEFFREY SHALLIT
University of Waterloo
Waterloo, ON, Canada
shallit@uwaterloo.ca

A *picture frame* in two dimensions is a rectangular array of symbols, with at least two rows and columns, where the first and last rows are identical, and the first and last columns are identical. If a coloring of the plane lattice has no picture frames, then we call it *frameless*. In this note, we show how to create a simple 2-coloring of the plane lattice that is frameless.

Before we get to the main result, let us talk about avoidance in one dimension.

Words and avoidance

Combinatorics on words is the area of discrete mathematics that deals with the properties of *words*: finite or infinite strings of symbols, typically defined over a finite alphabet Σ, such as $\Sigma = \{0, 1\}$.

A classic problem in the area is the avoidance of overlaps. An *overlap* is a word of the form $axaxa$, where a is a single symbol and x is a (possibly empty) word. The English word `alfalfa`, for example, forms an overlap with $a = \texttt{a}$ and $x = \texttt{lf}$. The Norwegian mathematician Axel Thue (1863–1922) proved that there exists a (one-sided) infinite word over a binary alphabet that contains no overlaps. We say that such a word is *overlap-free*. His classic paper, in German, was published in 1912 in an obscure Norwegian journal [**10**] and did not receive much attention until the 1980's. Thankfully, an English translation is available [**2**].

Today, the word that Thue constructed is called the *Thue-Morse word*, and it is often abbreviated **t**. Its first few terms are

$$\mathbf{t} = t(0)t(1)t(2)\cdots = 0110100110010110\cdots .$$

The Thue-Morse word **t** has many equivalent characterizations, but the following three are the most important:

(a) $t(2n) = t(n)$ and $t(2n+1) = \overline{t(n)}$ for $n \geq 0$. Here $\overline{0} = 1$ and $\overline{1} = 0$.

(b) There is a finite automaton of two states, which given n expressed in base 2 as an input, computes $t(n)$. This means that **t** is a 2-*automatic sequence* [**8**].

(c) $t(0)\cdots t(2^n - 1) = \mu^n(0)$, where μ is the morphism defined by $\mu(0) = 01$ and $\mu(1) = 10$ and the rule $\mu(xy) = \mu(x)\mu(y)$ for all words x, y. Here, the exponent on μ denotes n-fold composition of μ with itself. Thus we can write

$$\mathbf{t} = \mu^\omega(0) := \lim_{n\to\infty} \mu^n(0).$$

This expression is well-defined since $\mu(0)$ starts with 0, implying that $\mu^n(0)$ is a prefix of $\mu^{n+1}(0)$ for all n.

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Thue's original proof was neither very difficult nor very simple. Today, however, theorem-proving software can easily prove his result in less than a second. The idea is as follows: we write a formula in first-order logic that asserts that **t** is overlap-free. Next, we type this formula into `Walnut`—some wonderful software written by Hamoon Mousavi—that can prove or disprove any suitable first-order logical formula concerning automatic sequences [7]. Then we just have to read the result produced by the program!

If **t** has an overlap $axaxa$, then there exist integers $i \geq 0$ and $n = |ax| \geq 1$ such that $t(i + j) = t(i + n + j)$ for $0 \leq j \leq n$. So the nonexistence of overlaps is specified by the formula

$$\forall i, n\ (i \geq 0 \ \wedge\ n \geq 1) \implies (\exists j\ j \geq 0 \ \wedge\ j \leq n \ \wedge\ t(i + j) \neq t(i + n + j)).$$

A `Walnut` variable `T` encodes the Thue-Morse word. We can translate the formula above into a `Walnut` query named `tmpred` as follows:

```
eval tmpred "A i,n (n >= 1) => (Ej j<=n & T[i+j] != T[i+n+j])":
```

`Walnut` evaluates this formula as `true`, which provides the proof that the Thue-Morse word is overlap-free! (We do not have to include $i \geq 0$ and $j \geq 0$ in our `Walnut` formula because the domain of variables in its formulas is assumed to be $\mathbb{N}$ by default.)

Today, the field of pattern avoidance in words is extremely broad and dynamic, with many generalizations: avoidance in circular words [3], two-dimensional words [1], graphs [6], and so forth.

Thue himself proved a generalization of his result to "two-sided" infinite words. These are maps from $\mathbb{Z}$ to a finite alphabet Δ. We can turn a "right-infinite" infinite word into a "left-infinite" word with the reversal operator, which is denoted by an exponent of R. And we can concatenate a left-infinite word to a right-infinite word to produce a two-sided infinite word. Thue proved the following result:

Theorem 1. *The two-sided infinite word*

$$\mathbf{u} := \mathbf{t}^R\mathbf{t} = \cdots t(3)t(2)t(1)t(0)t(0)t(1)t(2)t(3)\cdots$$

is overlap-free.

Proof. The idea is that the "central" 2^{2n} bits of $\mathbf{t}^R\mathbf{t}$, namely

$$t(2^{2n-1} - 1)\cdots t(1)t(0)t(0)t(1)\cdots t(2^{2n-1} - 1),$$

equal $\mu^{2n}(1)$. Since $\mu^{2n}(1)$ is a suffix of $\mu^{2n+1}(0)$, it appears in **t**. Thus, an overlap appearing somewhere in $\mathbf{t}^R\mathbf{t}$ would imply the existence of exactly the same overlap in **t**, a contradiction.

To see that the central bits discussed above actually do equal $\mu^{2n}(1)$, note that by characterization (c) above, these central bits are $\mu^{2n-1}(0)^R\mu^{2n-1}(0)$. Next, let us show, by induction on e, that

$$\mu^e(a)^R = \begin{cases} \mu^e(\overline{a}), & \text{if } e \text{ odd;} \\ \mu^e(a), & \text{if } e \text{ even.} \end{cases} \tag{1}$$

for $e \geq 0$ and $a \in \{0, 1\}$. The base cases $e = 0, 1$ are easy. Otherwise, assume that equation (1) holds for $e - 1$, and let us prove it for e. Then

$$\mu^e(a)^R = \mu^{e-1}(a\,\overline{a})^R = (\mu^{e-1}(a)\,\mu^{e-1}(\overline{a}))^R = \mu^{e-1}(\overline{a})^R\mu^{e-1}(a)^R$$

$$= \begin{cases} \mu^{e-1}(\overline{a})\mu^{e-1}(a) = \mu^{e-1}(\overline{a}\,a) = \mu^e(\overline{a}), & \text{if } e \text{ is odd;} \\ \mu^{e-1}(a)\mu^{e-1}(\overline{a}) = \mu^{e-1}(a\,\overline{a}) = \mu^e(a), & \text{if } e \text{ is even.} \end{cases}$$

Thus, the central 2^{2n} bits of $\mathbf{t}^R\mathbf{t}$ are

$$\mu^{2n-1}(0)^R\mu^{2n-1}(0) = \mu^{2n-1}(1)\mu^{2n-1}(0) = \mu^{2n-1}(10) = \mu^{2n}(1),$$

and we are done. ■

Thue's two-sided infinite word $\mathbf{u}$ can alternatively be defined simply by extending the domain of t as follows: if n is negative, then define

$$t(n) := t(-1-n). \qquad (2)$$

Then $\mathbf{u} = \cdots t(-2)t(-1)t(0)t(1)t(2)\cdots$.

Pattern avoidance in two dimensions

Our goal is to give an example of pattern avoidance in the plane lattice $\mathbb{Z} \times \mathbb{Z}$. Although our result can be stated in the language of two-dimensional words, here it seems somehow more natural to use the language of colorings instead. A *coloring* of the plane lattice is a map f from $\mathbb{Z} \times \mathbb{Z}$ into a finite alphabet Δ, where the elements of Δ are called *colors*. If the number of elements of Δ is k, we call f a *k-coloring*.

The particular kind of pattern we avoid is called a "picture frame", or just "frame" for short. A *picture frame* is a finite rectangular array of points in the plane lattice, with at least two rows and columns, such that the first row equals the last row, and the first column equals the last column. Figure 1 illustrates part of a plane coloring having a picture frame in it. In this case, the alphabet Δ consists of English letters, and the highlighted frame is bordered by the horizontal word `ERMINE` and the vertical word `EWE`.

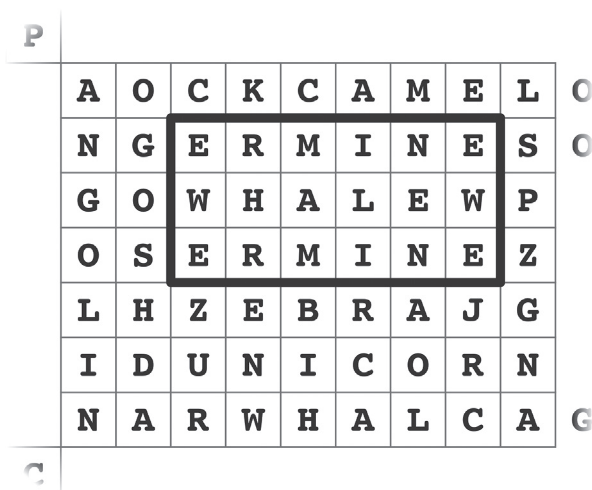

Figure 1 A portion of a plane coloring that contains a picture frame.

Now the following natural question occurs: is there a k-coloring of the plane lattice containing no picture frames at all? We might call such a coloring *frameless*.

At first, one might suspect the answer is no, since that is the case for the one-dimensional analogue of picture frames. The one-dimensional analogue is a word of the form axa, where a is a single letter, over a k-letter alphabet. By the pigeonhole principle, every word of length at least $k+1$ contains a repeated letter, and hence a word of the form axa. So there are no infinite words without 1-dimensional picture frames.

Furthermore, Gallai's theorem (see, for example, Graham et al. [4]) implies that any finite coloring of $\mathbb{N} \times \mathbb{N}$ contains a square subarray with all four corners colored identically.

Nevertheless, in this note we prove the following result:

Theorem 2. *There exists a frameless* 2*-coloring of the infinite plane lattice.*

The coloring and the proof

We start by proving the result for the quarter plane $\mathbb{N} \times \mathbb{N}$.

Theorem 3. *For $i, j \geq 0$ define the coloring f by $f(i, j) = t(i + j)$, where $\mathbf{t} = t(0)t(1)t(2)\cdots$ is the Thue-Morse sequence. Then f is a frameless* 2*-coloring of $\mathbb{N} \times \mathbb{N}$.*

Here are the first few rows and columns of this coloring, where 0 is denoted by a white square and 1 by a black square. The coloring is easily seen to consist of shifted copies of the one-dimensional Thue-Morse word.

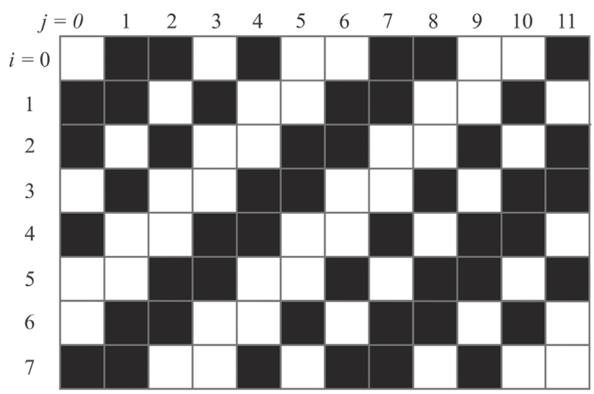

Figure 2 A portion of a frameless 2-coloring.

There are at least two ways to prove the result. One uses the theorem-proving software `Walnut` again:

First proof of Theorem 3. Let us write a first-order statement for the nonexistence of a picture frame in $(f(i, j))_{i,j\geq 0}$:

$$\neg(\exists m, n, p, q\ (p \geq 1)\ \wedge\ (q \geq 1) \wedge$$
$$(\forall i\ (i \leq q) \implies f(m, n+i) = f(m+p, n+i)) \wedge$$
$$(\forall j\ (j \leq p) \implies f(m+j, n) = f(m+j, n+q)).$$

Here the clause

$$(\forall i\ (i \leq q) \implies f(m, n+i) = f(m+p, n+i))$$

asserts that the top and bottom sides of the picture frame are the same, and the clause

$$(\forall j\ (j \leq p) \implies f(m+j, n) = f(m+j, n+q))$$

asserts that the left and right sides of the picture frame are the same.

Using the fact that $f(i, j) = t(i + j)$, this can be translated into `Walnut` as follows:

```
eval frameless "~E m,n,p,q (p>=1) & (q>=1) & (Ai (i<=q) =>
```

```
T[m+n+i]=T[m+p+n+i]) & (Aj (j<=p) => T[m+j+n]=T[m+j+n+q]))":
```

Evaluating this statement in Walnut gives the response true, and requires less than a second of CPU time on a laptop. ■

Alternatively, relying on the fact that **t** is overlap-free, we can prove Theorem 3 directly. We use the abbreviation $f(a..b, c..d)$ to denote the rectangular block with rows from a to b and columns from c to d.

Second proof of Theorem 3. Suppose there exist m, n, p, q such that $p, q \geq 1$ and

$$f(m, n..n+q) = f(m+p, n..n+q)$$

and

$$f(m..m+p, n) = f(m..m+p, n+q).$$

Then, since $f(i, j) = t(i+j)$, we have

$$t(x+i) = t(x+p+i),\ 0 \leq i \leq q,$$

$$t(x+j) = t(x+q+j),\ 0 \leq j \leq p,$$

where $x = m+n$. Without loss of generality, assume $p \leq q$. But then

$$t(x)t(x+1)\cdots t(x+2p) = azaza$$

with $a = t(x)$ and

$$z = t(x+1)\cdots t(x+p-1) = t(x+p+1)\cdots t(x+2p-1).$$

This shows that **t** has the overlap $azaza$, which is the desired contradiction. ■

There is another way to view this result, using the notion of *two-dimensional morphisms* [9]. These are morphisms h that map each letter a to a $k \times \ell$ rectangular block of letters. If x is an $m \times n$ matrix, then $h(x)$ is a $km \times \ell n$ matrix.

Now consider the two-dimensional morphism γ defined as follows:

$$\gamma(0) = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix},$$

$$\gamma(1) = \gamma(2) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\gamma(3) = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix},$$

and the morphism τ defined by $\tau(i) = i \bmod 2$.

Theorem 4. $(f(i, j))_{i,j\geq 0}$ *is given by* $\tau(\gamma^{\omega}(0))$.

Proof. (Sketch) The basic idea is that if we have the values of $t(x)$ and $t(x+1)$, for $x = m+n$, then the values of $f(2m+a, 2n+b)$ for $a, b \in \{0, 1\}$ can be computed as follows:

$$f(2m, 2n) = t(x),$$

$$f(2m+1, 2n) = f(2m, 2n+1) = \overline{t(x)},$$

$$f(2m+1, 2n+1) = t(x+1).$$

For example

$$f(2m, 2n) = t(2m+2n) = t(m+n) = t(x),$$

where we have used characterization (a) of **t**. ■

Let us look at the first few iterates of γ, and their images under τ:

$$[0], \quad \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 3 & 0 & 3 \\ 3 & 0 & 3 & 2 \\ 0 & 3 & 2 & 0 \end{bmatrix}, \quad \ldots$$

$$[0], \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \ldots.$$

The arrays in the second line form larger and larger portions of the upper left corner of the array in Figure 2. This observation will be useful in the next section.

Extending our coloring to the whole plane

Now that we have a coloring of $\mathbb{N} \times \mathbb{N}$, we can extend it to $\mathbb{Z} \times \mathbb{Z}$, and prove the following explicit version of Theorem 2:

Theorem 2′. *Using the extended definition of t given in Eq. (2), define $f(i, j) = t(i + j)$ for $i, j \in \mathbb{Z}$. Then $(f(i, j))_{i,j\in\mathbb{Z}}$ is frameless.*

Proof. Exactly the same as the proof of Theorem 3, using the fact that the two-sided infinite word **u** is overlap-free. ■

A slightly different example of a frameless 2-coloring of $\mathbb{Z} \times \mathbb{Z}$ can be constructed using the matrix-valued morphism γ and the morphism τ defined above. To do so, we start with the 2×2 array $M := \gamma(0)$ and consider the center of the array to be a point from which all four quarter-planes extend outward, as in Figure 3:

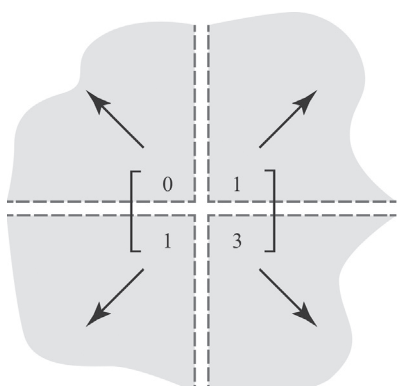

Figure 3 Iterating a 2-D morphism to cover the plane.

We then want to iterate γ on each block. In order for this procedure to work, we would need the image of each letter a to match the letter in the corresponding corner:

- 0 should appear in the lower right of its image;
- 1 should appear in the upper right and lower left of its image; and
- 3 should appear in the upper left of its image.

These properties do not hold for γ. But they do hold for γ^2! We have

$$\gamma^2(0) = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 3 & 0 & 3 \\ 3 & 0 & 3 & 2 \\ 0 & 3 & 2 & 0 \end{bmatrix}, \quad \gamma^2(1) = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 2 \\ 1 & 3 & 2 & 0 \end{bmatrix}, \quad \gamma^2(3) = \begin{bmatrix} 3 & 2 & 0 & 3 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 3 \end{bmatrix}.$$

Applying γ^2 to M gives

$$\begin{array}{cccc|cccc} 0 & 1 & 3 & 0 & 3 & 2 & 0 & 1 \\ 1 & 3 & 0 & 3 & 2 & 0 & 1 & 3 \\ 3 & 0 & 3 & 2 & 0 & 1 & 3 & 2 \\ 0 & 3 & 2 & 0 & 1 & 3 & 2 & 0 \\ \hline 3 & 2 & 0 & 1 & 3 & 2 & 0 & 3 \\ 2 & 0 & 1 & 3 & 2 & 0 & 3 & 0 \\ 0 & 1 & 3 & 2 & 0 & 3 & 0 & 1 \\ 1 & 3 & 2 & 0 & 3 & 0 & 1 & 3 \end{array}$$

Hence, by iterating γ infinitely, and then applying τ, we get a 2-coloring of the entire plane with the desired property.

Connection to aperiodic tilings

Suppose we have some finite set of shapes. A *tiling of the plane* is an arrangement of translated, rotated, and reflected copies of these shapes that completely covers the plane with no gaps and no overlaps.

A *periodic tiling* is a tiling of the plane that has two linearly independent translation symmetries: two different directions in which you can slide the tiling and have all the shapes line up perfectly with translated copies. There are periodic tilings of the plane by squares and regular hexagons, for example.

A set of shapes is called *aperiodic* when it permits tilings of the plane, but none that are periodic. Tilings that are not periodic are not necessarily interesting of their own accord because they might be constructed from shapes that can trivially be rearranged into periodic tilings. The interesting case is when the shapes themselves *force* long-range aperiodicity, which is why aperiodicity is a property ascribed to the shapes, known as *prototiles*. The most famous aperiodic tile sets are those discovered independently by Penrose and Ammann [**5**].

Consider a tiling whose prototiles are polyominoes (unit squares glued together along their edges). Every member of the set appears in one of (at most) eight rotated and reflected orientations in the tiling. Assign the integers $\{1, \ldots, n\}$ to the squares that make up all prototiles. Then, every cell in a tiling using those prototiles can be given a label from $\{1, \ldots, n\} \times \{1, \ldots 8\}$, describing the identity and orientation of the prototile square covering that cell. In other words, a tiling by polyominoes is also a coloring of the plane lattice. Figure 4 shows two annotated prototiles on the left, and a portion of a tiling/coloring on the right.

We know that there exist aperiodic sets of polyominoes—Winslow offers one example adapted from an earlier tiling by Ammann [**11**]. The existence of such prototiles suggests an indirect proof that frameless colorings of the plane lattice must exist. Consider a coloring constructed by applying the labelling method above to a tiling by an aperiodic set of polyominoes. If a frame existed in such a coloring, then we would be able to repeat it in a rectangular arrangement, with adjacent horizontal and vertical copies overlapping by one column and one row, respectively. The periodic coloring thus obtained would imply a periodic tiling by the original polyominoes, contradicting the assumption that they were aperiodic.

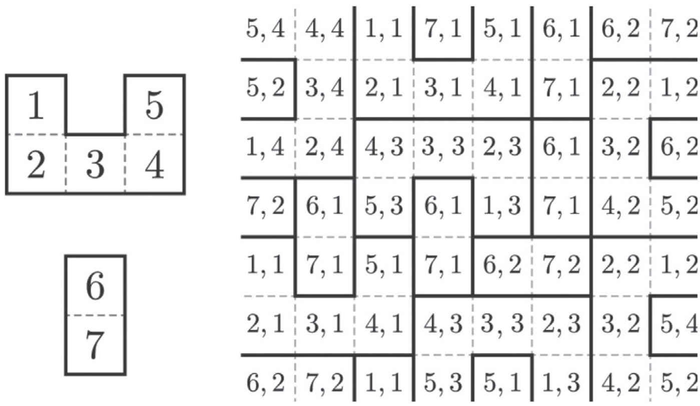

Figure 4 Prototiles and tiling.

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Summary. A *picture frame* in two dimensions is a rectangular array of symbols, with at least two rows and columns, where the first and last rows are identical, and the first and last columns are identical. If a coloring of the plane lattice has no picture frames, we call it *frameless*. In this note, we show how to create a simple 2-coloring of the plane lattice that is frameless.

CRAIG S. KAPLAN (MR Author ID: 683984) is a professor of computer science at the University of Waterloo. He received a B. Math. in pure mathematics and computer science from Waterloo, and an M. S. and Ph.D. in computer science from the University of Washington. He studies applications of computer graphics and mathematics to problems in art, architecture and design. He is an Associate Editor of the *Journal of Mathematics and the Arts*, and helps to organize the annual Bridges conference on art and mathematics.

JEFFREY SHALLIT (MR Author ID: 159555) is a professor of computer science at the University of Waterloo. He received an A. B. in mathematics from Princeton University and a Ph.D. in mathematics at the University of California, Berkeley. He has published approximately 275 papers on number theory, algebra, algorithms, combinatorics, computational complexity, formal languages, automata theory, and the history of mathematics. His Erdős number is 1 and his Elvis number is 3.

Harriot's Observation of Resisted Trajectories

C. W. GROETSCH
The Citadel
Charlestown, SC 29409
charles.groetsch@citadel.edu

For nearly two millennia, Aristotle's theory of violent and natural local motion held sway. Such was the influence of the great philosopher that the notion of a hyper-angular ballistic trajectory, reminiscent of a Wile E. Coyote cartoon, consisting of a violent linear motion, followed by a vertical natural motion, was taken seriously well into the sixteenth century (see Figure 1). Early in the sixteenth century, mathematicians, notably Niccolo Tartaglia (1499-1557), endeavored to formulate theories to better describe actual trajectories observed by gunners for several centuries.

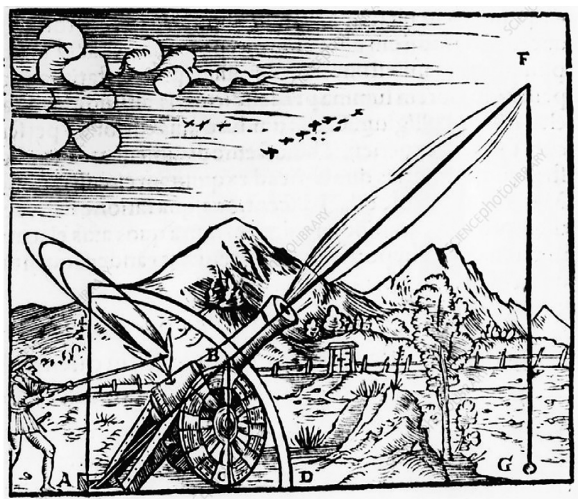

Figure 1 An Aristotelian Ttrajectory. The image is from Santbech [**8**].

Tartaglia's proposed trajectory modified the Aristotelian trajectory by blending a transitional arc between straight line violent and natural motions, see Figure 2 [**12**, p. 40]. In his words (translation from [**12**]; [**1**, p. 84]), "Every violent trajectory or motion ... will always be partly straight and partly curved, and the curved part will form part of the circumference of a circle. Yet in the very next sentence he continued, "Truly no violent trajectory or motion ... can have any part that is perfectly straight ..., indicating that he realized his trajectory was an idealization.

The work of Thomas Harriot (1560–1621) on trajectories is not well-known today because he published nothing during his lifetime, other than his *A Briefe and True Report of the New Found Land of Virginia* (1588). His nachlass, comprising thousands of folio sheets containing notes on algebra, astronomy, navigation, mechanics, optics and other subjects, resided in private hands for more than three centuries. Around 1784, the bulk of these papers came under the control of Franz Zach who, in the employ of Lord Egremont

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> . . . had gone through them so hurriedly, seeking sensational or exciting materials, and had been careless about retaining their original order. Much of the time he had turned the pages as he read them; at other times he piled the pages without turning them over, and in some instances he had gathered bundles without regard for top or bottom or had put together sections which did not belong together. As a result . . . Harriot's papers were in an almost chaotic condition. [10, p. 20]

Thanks largely to the intrepid scholars Johannes Lohne [6] and Martin Schemmel [9], Harriot's notes have been disentangled and interpreted.

In his unpublished notes, a sketch in Harriot's hand can be seen of a typical trajectory familiar to contemporary practitioners of "great artillery." This sketch indicates that Harriot turned away from the tripartite violent-transitional-natural Tartaglian trajectory in favor of a bipartite trajectory consisting of two branches, neither of which contain any straight part. The ascending branch of the trajectory rises to a "culmination point" (apex), followed by a descending branch ending at ground level. Schemmel's comment that the ascending branch is "longer and flatter than the path from the culmination point down to the ground" [9, p. 28] is the inspiration for this note.

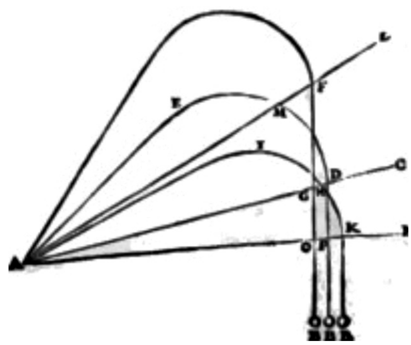

Figure 2 Tartaglian Trajectories.

Studies of projectile motion subjected to resistance proportional to velocity are a rich source of analytical topics accessible to undergraduates. Several recent works on this theme have dealt with quantitative aspects of trajectory optimization and some have reintroduced and popularized the Lambert W-function (for example, Kantrowitz and Neumann [5], and Packel and Yuen [7]). Other articles have made connections with the history of science and have emphasized qualitative aspects of trajectories ([2], [3], [4], [11]). This note, which is of the latter sort, is an entirely elementary treatment validating Schemmel's remark on the relative length and flatness of the two branches of a linearly resisted trajectory. We interpret and validate the flatness and length observation, from both a macro and micro perspective, for a simple model of projectile motion in a resisting medium. Our analysis originated in discussions in an honors calculus course and uses only basic concepts and techniques that are accessible to first-year calculus students.

The model

Newton's laws of motion for a point particle of unit mass launched from the origin into a medium offering resistance proportional to velocity, with initial speed v, at an angle

$\theta \in [0, \pi/2)$ with respect to the positive x-axis, are expressed by the system of initial value problems

$$\ddot{x} = -k\dot{x},\ x(0) = 0,\ \dot{x}(0) = v\cos\theta,$$
$$\ddot{y} = -g - k\dot{y},\ y(0) = 0,\ \dot{y}(0) = v\sin\theta,$$

where $k > 0$ is a drag coefficient and g is the constant acceleration of gravity. The dots signify derivatives with respect to time t. Integrating this un-coupled system reveals the parameterized trajectory

$$x(t) = \frac{v\cos\theta}{k}\left(1 - e^{-kt}\right),$$
$$y(t) = \frac{g}{k^2}\left(1 + \frac{vk\sin\theta}{g}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t. \tag{1}$$

On setting

$$z = \frac{k}{v\cos\theta}x = 1 - e^{-kt}$$

and eliminating the parameter t, we find that the shape of the trajectory may be expressed by the shorter expression

$$u(z) = cz + \ln(1 - z), \quad 0 \le z < 1, \tag{2}$$

where $u = (k^2/g)y$ and the parameter

$$c = 1 + \frac{kv\sin\theta}{g} > 1$$

encapsulates all physical parameters (g, k) and control parameters (v, θ) of the model. Furthermore, since

$$\frac{dz}{dx} = \frac{k}{v\cos\theta},$$

we have

$$\frac{du}{dz} = \frac{k^2}{g}\frac{dy}{dz} = \frac{k^2}{g}\frac{dy}{dx}\frac{dx}{dz} = \frac{kv\cos\theta}{g}\frac{dy}{dx},$$

showing that the derivatives $u'(z)$ and $y'(x)$ are, for fixed values of the parameters, positively proportional. Therefore, qualitative notions of "flatness" are the same for either of the descriptions in equations (1) and (2).

Basic features of the graph of function (2) are easily verified: it achieves its maximum at the point

$$(z^*, u^*) = (1 - 1/c,\ c - 1 - \ln c), \tag{3}$$

it is strictly increasing on the interval $(0, z^*)$ and strictly decreasing on $(z^*, 1)$, and it is asymptotic to the line $z = 1$. So, in addition to the root at the origin, u has a unique root R in the interval $(z^*, 1)$. Also, $u'(z) = c - 1/(1 - z)$ is not constant on any subinterval of $(0, R)$, reflecting Tartaglia's comment that his trajectory "can have no part that is perfectly straight," in sharp contrast to the Aristotelian trajectory of Figure 1. The graph of a typical function in the class (2) appears in Figure 3.

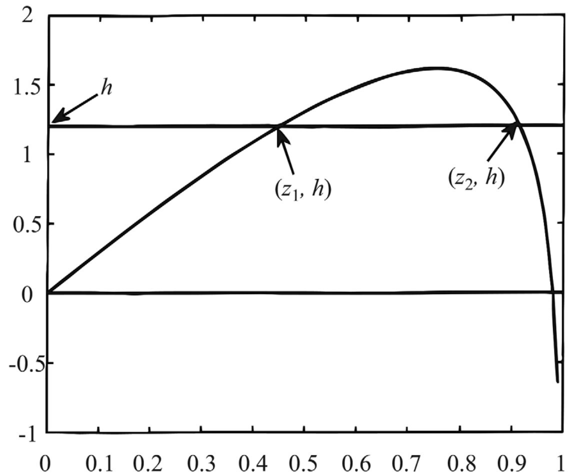

Figure 3 A generic trajectory.

It is apparent from this graph that resistance causes a certain asymmetry: z^* lies beyond the mid-range $R/2$. This fact, proved in Groetsch [3] by a timing-related argument, is a consequence of the more general result given in equation (5) below.

The intermediate value theorem and the strict monotonicity of the ascending, respectively descending, branches of equation (2) guarantee that for a given number $h \in [0, u^*)$ there is a unique $z_1 = z_1(h) \in [0, z^*)$ and a unique $z_2 = z_2(h) \in (z^*, R)$ satisfying

$$u(z_1) = h = u(z_2). \tag{4}$$

By extension we define $z_1(u^*) = z_2(u^*) = z^*$.

As h traverses the interval $[0, u^*)$ of the vertical axis, the horizontal lines through $(0, h)$ rise, the pre-image $z_1(h)$ advances, and $z_2(h)$ retreats. It is less apparent that as h increases the average value of the pre-images remains strictly less than z^*.

Proposition 1. *If $h \in [0, u^*)$, then*

$$\frac{z_1(h) + z_2(h)}{2} < z^*. \tag{5}$$

Proof. The proof is accomplished by sequential reformulation of equation (5) and a change of variable. To streamline notation, $z_1(h)$ will be represented by z_1 and similarly $z_2 = z_2(h)$.

First, note that equation (5) is equivalent to $z_1 < p$, where $p = 2z^* - z_2 < z^*$. Also, since $u(z)$ is strictly increasing for $z < z^*$ and strictly decreasing for $z > z^*$, and since

$$u(z_2) = h = u(z_1),$$

we have that $z_1 < p$ is equivalent to $u(z_2) < u(p)$. That is,

$$c(z_2 - p) + \ln(1 - z_2) < \ln(1 - p),$$

or equivalently,

$$c(1 - z_2)e^{c(z_2 - p)} < c(1 - p). \tag{6}$$

However, by equation (3),

$$c(1 - p) = c(1 + z_2 - 2z^*) = 2c(1 - z^*) - c(1 - z_2)$$

$$= 1 + (1 - c(1 - z_2)),$$

and hence

$$c(z_2 - p) = c(z_2 - 1 + 1 - p) = 2(1 - c(1 - z_2)).$$

Setting $w = 1 - c(1 - z_2)$, we note that $w < 1$ and, since $z_2 > z^*$, we see that $w > 1 + c(z^* - 1) = 0$. Making this substitution and rearranging a bit, inequality (6) becomes:

$$f(w) = (1 - w)[1 + e^{2w}] < 2, \quad 0 < w < 1.$$

But note that $f(0) = 2$, and for $w > 0$ we have

$$f'(w) = e^{2w}(1 - 2w) - 1 < e^{2w}e^{-2w} - 1 = 0.$$

Hence, $f(w) < 2$, implying f is strictly deceasing, proving (6) and thereby establishing equation (5). ■

This result is key to comparing the relative flatness of the ascending and descending branches of the trajectory. In particular we note that

$$\frac{z_1(0) + z_2(0)}{2} = \frac{R}{2} < z^*. \tag{7}$$

Flatness and length

How is the "flatness" of a branch to be interpreted? A reasonable macroscopic gauge is the average flatness—the average slope of the tangent line. We now show that the average value of $u'(z)$ on the ascending branch, $u(z)$, $0 \le z \le z^*$, is strictly less than the average value of $-u'(z)$ on the descending branch. That is, for $z^* \le z \le R$. This gives a global interpretation of the observation that the trajectory is *flatter* to the left of the "culmination point" (z^*, u^*) than it is to the right of that point.

The average value of $u'(z)$ on the ascending branch is

$$A = \frac{1}{z^*}\int_0^{z^*} u'(z)\,dz = \frac{u^*}{z^*},$$

while the average value of $-u'(z)$ on the descending branch is

$$D = \frac{-1}{R - z^*}\int_{z^*}^{R} u'(z)\,dz = \frac{u^*}{R - z^*}.$$

Therefore, $A < D$ if and only if $R < 2z^*$, a fact established in equation (7) of the previous section.

A natural microscopic, or pointwise, interpretation of relative flatness is given in terms of the h-level sets

$$\{z : u(z) = h \in [0, u^*)\},$$

each of which consists of exactly two points, one on the ascending branch, namely $z_1(h)$, and the other, $z_2(h)$, on the descending branch.

We show that

$$\frac{du}{dz}(z_1) < -\frac{du}{dz}(z_2),$$

expressing the point-wise relative "flatness" of the ascending branch. To this end, it is helpful to note the following:

Lemma 1. *The function*

$$g(h) = \frac{1}{1 - z_1(h)} + \frac{1}{1 - z_2(h)}$$

is strictly decreasing on the interval $(0, u^*)$.

Proof. This is a consequence of equation (5).

In fact, from equation (4) we find that

$$\left(c - \frac{1}{1 - z_i}\right)\frac{dz_i}{dh} = 1, \quad i = 1, 2,$$

and hence that

$$\begin{aligned} g'(h) &= (1 - z_1)^{-2}\frac{dz_1}{dh} + (1 - z_2)^{-2}\frac{dz_2}{dh} \\ &= \frac{1}{(1 - z_1)(c(1 - z_1) - 1)} + \frac{1}{(1 - z_2)(c(1 - z_2) - 1)}. \end{aligned}$$

Since $1 - z_1 > 1 - z^*$ and $0 < 1 - z_2 < 1 - z^*$, we see that

$$c(1 - z_1) - 1 > c(1 - z^*) - 1 = 0$$

and

$$c(1 - z_2) - 1 < c(1 - z^*) - 1 = 0,$$

respectively. From this and equation (3), we obtain

$$\begin{aligned} g'(h) &< \frac{1}{(1 - z^*)(c(1 - z_1) - 1)} + \frac{1}{(1 - z^*)(c(1 - z_2) - 1)} \\ &= \frac{c}{c(1 - z_1) - 1} + \frac{c}{c(1 - z_2) - 1} = \frac{1}{z^* - z_1} + \frac{1}{z^* - z_2}. \end{aligned}$$

Finally, equation (5) gives $z_2 - z^* < z^* - z_1$ and hence $1/(z_2 - z^*) > 1/(z^* - z_1)$. Therefore,

$$g'(h) < \frac{1}{z^* - z_2} + \frac{1}{z_2 - z^*} = 0,$$

proving that $g(h)$ is strictly decreasing. ■

Proposition 2. *For each* $h \in (0, u^*)$, *we have that*

$$\frac{du}{dz}(z_1) < -\frac{du}{dz}(z_2).$$

Proof. By equation (2), the desired inequality is the same as

$$c - \frac{1}{1 - z_1(h)} < -c + \frac{1}{1 - z_2(h)},$$

or equivalently,

$$2c < \frac{1}{1 - z_1(h)} + \frac{1}{1 - z_2(h)},$$

that is, $2c < g(h)$. But $g(h)$ is strictly decreasing on $(0, u^*)$, as established in the lemma. Note that

$$g(u^*) = \frac{2}{1 - z^*} = 2c.$$

We claim that

$$g(0) = \frac{1}{1 - 0} + \frac{1}{1 - R} > 2c,$$

or equivalently,

$$\frac{2c - 2}{2c - 1} < R. \tag{8}$$

Since $0 < (2c - 2)/(2c - 1) < 1$, we see that equation (8) holds if and only if

$$u\left(\frac{2c - 2}{2c - 1}\right) > u(R) = 0.$$

Setting $x = 2c - 1$, we find that $x > 1$ and

$$u\left(\frac{2c - 2}{2c - 1}\right) = u\left(\frac{x - 1}{x}\right) = \frac{x + 1}{2}\frac{x - 1}{x} - \ln x.$$

Hence, the required condition (8) is equivalent to

$$x \ln x < \frac{x^2 - 1}{2} \quad \text{for} \quad x > 1. \tag{9}$$

Since $1/t < 1$ for $t > 1$, we see that for $x > 1$,

$$\int_1^x \ln s \, ds < \int_1^x \int_1^s \frac{1}{t} \, dt \, ds < \int_1^x \int_1^s 1 \, dt \, ds = \int_1^x s - 1 \, ds.$$

That is,

$$x \ln x - x + 1 < x^2/2 - x + 1/2,$$

proving equation (9) and verifying that $g(0) > 2c$. As $g(h)$ is strictly decreasing on $(0, u^*)$ and $g(u^*) = 2c$, we find that $g(h) > 2c$ for all $h \in [0, u^*)$.

Therefore, at each level $h \in [0, u^*)$

$$\frac{du}{dz}(z_1) < -\frac{du}{dz}(z_2). \qquad \blacksquare$$

That the ascending branch is longer than the descending branch is an easy consequence of this "flatness" result. In fact, given $z \in [0, z^*]$ there is a unique $y \in [z^*, R]$ with $u(z) = u(y)$ (in the former notation $y = z_2(u(z))$). Differentiating with respect to z, we obtain

$$\frac{dy}{dz} = \frac{u'(z)}{u'(y)} \quad \text{for} \quad z \in [0, z^*).$$

Therefore,

$$\left(\frac{dy}{dz}\right)^2 < 1. \tag{10}$$

Noting that y decreases with increasing z, and changing variable in the integral representing arclength of the descending branch, we find that:

$$\int_{z^*}^{R} \sqrt{1+u'(y)^2}\,dy = \int_{z^*}^{0} \sqrt{1+u'(y)^2}\,\frac{dy}{dz}\,dz = \int_0^{z^*} \sqrt{1+u'(y)^2}\,(-\frac{dy}{dz})\,dz$$

$$= \int_0^{z^*} \sqrt{1+u'(y)^2}\,|\frac{dy}{dz}|\,dz = \int_0^{z^*} \sqrt{(\frac{dy}{dz})^2 + (\frac{du}{dy}\frac{dy}{dz})^2}\,dz.$$

Finally, by equation (10),

$$\int_{z^*}^{R} \sqrt{1+u'(y)^2}\,dy = \int_0^{z^*} \sqrt{\left(\frac{dy}{dz}\right)^2 + \left(\frac{du}{dy}\frac{dy}{dz}\right)^2}\,dz < \int_0^{z^*} \sqrt{1+u'(z)^2}\,dz.$$

That is, the ascending branch is longer than the descending branch.

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Summary. Features of Thomas Harriot's sketch of the shape of cannon ball trajectories, commonly observed by late sixteenth century gunners, are validated for a model of linearly resisted projectiles. The analysis uses only basic concepts and familiar techniques from first-year calculus.

C. W. GROETSCH (MR Author ID: 77275) is Distinguished Professor of Mathematical Science at The Citadel in Charleston, SC, where he was the founding dean of the School of Science and Mathematics. Currently one of his interests is the history of the mathematization of physical science in the sixteenth and seventeenth centuries.

The Grazing Goat and Spherical Curiosities

MICHAEL HARRISON
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
mah5044@gmail.com

The following problem appeared in the very first volume of the *American Mathematical Monthly*, in 1894 [**19**].

Grazing Goat Problem *A goat grazing in a unit-circular field is tethered by rope to a fence enclosing the field. What length of rope ensures that the goat can graze exactly half of the field?*

This is illustrated in the left side of Figure 1 .

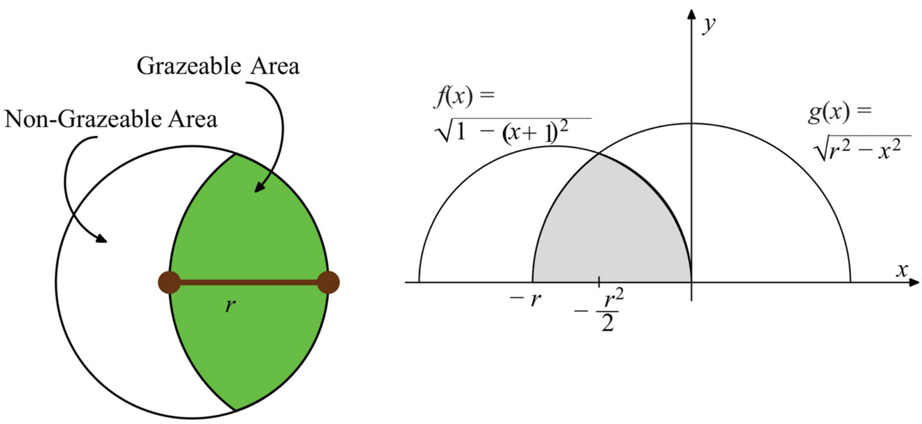

Figure 1 Two depictions of the grazing goat problem.

The grazing goat has engrossed mathematicians for over a century, evidenced by its appearances in Fraser [**9**, **10**], Hoffman [**11**], and the eighteen additional articles referenced in [**9**]. One alluring aspect of the problem is its deceptive difficulty. Indeed, attempted solutions to the grazing goat problem yield, at best, the desired rope length embedded implicitly in the clutches of trigonometric functions or radicals, so that the use of approximation procedures (or suitable mathematical software) is necessary to compute the solution length of approximately 1.15872847. Nevertheless, an implicit solution can be teased out using only trigonometric techniques, and we encourage the reader to attempt this approach before continuing. Those having difficulty might consult Figure 4 for a hint or Fraser [**9**] for a full (and eloquently delivered) solution.

The right side of Figure 1 suggests calculus as a suitable alternative to trigonometry. We first let the graph of

$$f(x) = \sqrt{1-(x+1)^2}$$

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MSC: 53A05

represent the top half of the goat's pen. If we tie the goat with a rope of length r to the rightmost part of the fence, then the portion of the graph of

$$g(x) = \sqrt{r^2 - x^2}$$

which lies inside the pen represents the boundary of the penned area accessible to the goat. Solving the equation $f(x) = g(x)$ gives the x-coordinate x_I of the intersection point of the two curves: $x_I = -\frac{r^2}{2}$. The total grazeable area $A(r)$ is twice the shaded area in the right side of Figure 1, and so we have

$$A(r) = 2\int_{-r}^{-r^2/2} \sqrt{r^2 - x^2}\, dx + 2\int_{-r^2/2}^{0} \sqrt{1 - (x+1)^2}\, dx. \tag{1}$$

The grazing goat problem asks which length of rope r_0 yields a grazeable area equal to half of the total area π of the (unit circular) pen. To find r_0 satisfying $A(r_0) = \pi/2$, one may first use trigonometric substitutions to evaluate the integrals in equation (1). The resulting expression may be simplified, so that the condition $A(r_0) = \pi/2$ reduces to the condition

$$0 = \frac{\pi}{2} + (r^2 - 2)\arccos\frac{r}{2} - \frac{r}{2}\sqrt{4 - r^2}. \tag{2}$$

The reader is free to work out the details, but this is not a requirement for understanding later sections. In fact, the path from (1) to (2) is considerably more cumbersome than a direct trigonometric solution, and we much prefer the latter approach.

Instead, we will focus on higher-dimensional variants of the grazing goat problem. In our study, we will find connections to certain surprising properties of spheres—including the curious volume-accumulation of high-dimensional spheres and the Hat-Box (or Equal Area Zones) Property of the 2-sphere, originally due to Archimedes. This note may be seen as a tour through these peculiar properties, with the grazing goat leading the way.

The grazing goat in three dimensions

A higher-dimensional analogue of the grazing goat problem was formulated and studied several decades ago by Fraser [**10**]. In 3-dimensional Euclidean space, a physical description of this problem is still readily attainable: consider a bird in a unit spherical cage, attached (carefully!) to the cage with some length of string. (Fraser actually preferred imagining a tried-and-true goat nibbling its way through a grassy sphere). One may picture a bird free to fly about the shaded region in the picture on the left side of Figure 2. Fraser asked: *What length of string allows the bird to roam a region with volume equal to exactly half of the total volume of the cage?* This problem can be studied in the language of surfaces of revolution: starting with the picture on the right side of Figure 2 (which is identical to the right side of Figure 1), we revolve the graphs of $f(x)$ and $g(x)$ about the x-axis to produce the picture on the left side.

Using Figure 2 for guidance, we see that the shaded region on the left side is composed of two solid spherical caps, obtained

1. by revolving the area under the graph of $g(x)$, $-r \le x \le -\frac{r^2}{2}$, and
2. by revolving the area under the graph of $f(x)$, $-\frac{r^2}{2} \le x \le 0$.

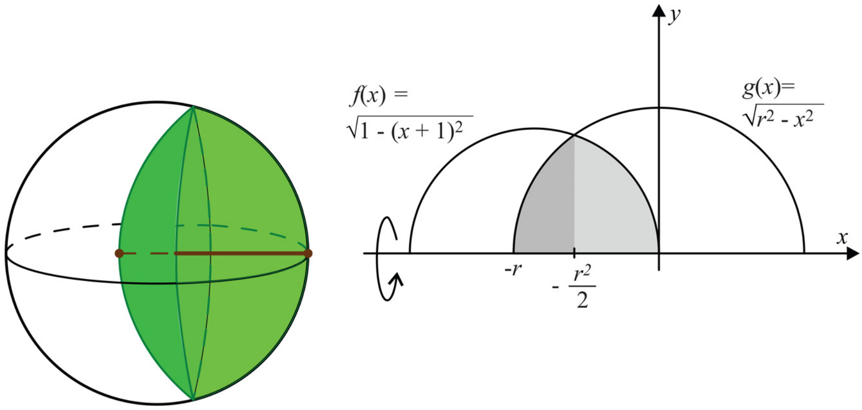

Figure 2 The grazing goat in $\mathbb{R}^3$ – the left picture is obtained by revolving the right picture

Insisting that the sum of these volumes equals half of the cage volume yields the equation

$$\int_{-r}^{-r^2/2} \pi g(x)^2\, dx + \int_{-r^2/2}^{0} \pi f(x)^2\, dx = \frac{2\pi}{3}. \tag{3}$$

In this case, both integrals may be integrated directly to yield the polynomial equation

$$3r^4 - 8r^3 + 8 = 0.$$

There are no rational solutions, but we can approximate the solution ≈ 1.228545 using software or our favorite root-finding method. In fact, this 3-dimensional problem is a neat exercise for students familiar with solids of revolution and with Newton's method.

The grazing goat in higher dimensions

Fraser continues to higher dimensions by posing the following n-dimensional grazing goat problem: A goat grazing in an n-dimensional field is tethered to the spherical fence enclosing the field. What length of rope ensures that the goat can graze exactly half of the field?

Although we sacrifice a physical interpretation in higher dimensions, the problem is mathematically feasible. For full details, we refer the reader to Fraser's original exposition. With our own goat problems to solve, we only summarize Fraser's findings.

In each dimension, Fraser arrives at an equation which implicitly defines the solution length. Table 1 gives the resulting equations for small dimensions, as well as an approximation of each solution length r_n. Fraser also provides evidence that $r_n \to \sqrt{2}$ as $n \to \infty$, but his proof contains an error, noticed and reconciled by Meyerson [14]. Towards the end of this paper, we provide a new geometric perspective for this asymptotic behavior.

A variation of the grazing goat problem

It would hardly surprise us to discover that goats dislike being tethered. This motivates us to consider a more humane idea: in lieu of tying the goat to the pen, we build a

n	Implicitly defined rope length	Solution length
2	$0 = \frac{\pi}{2} + (r^2 - 2)\arccos\frac{r}{2} - \frac{r}{2}\sqrt{4 - r^2}$	$r_2 \approx 1.158728$
3	$0 = 3r^4 - 8r^3 + 8$	$r_3 \approx 1.228545$
4	$0 = 3\pi + (6r^4 - 12)\arccos\frac{r}{2} - \frac{2r^5 + r^3 + 6r}{2}\sqrt{4 - r^2}$	$r_4 \approx 1.268079$
5	$0 = 5r^8 - 80r^6 + 128r^5 - 128$	$r_5 \approx 1.293598$

TABLE 1: Solutions to the n-dimensional grazing goat problem in low dimensions

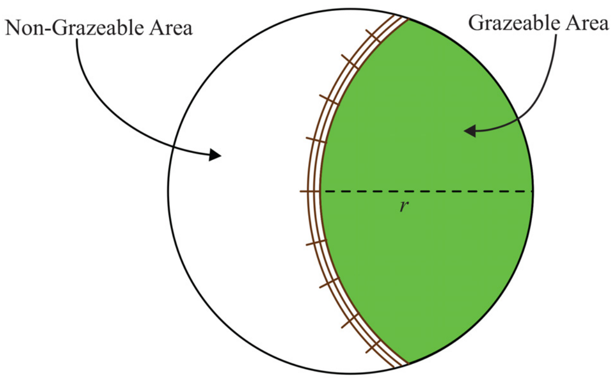

Figure 3 A fence to replace the rope.

fence in the shape of a circular arc to fence off the area where the goat can roam (see Figure 3). We replace the requirement that the grazeable area is half of the total area, and instead *we seek the radius of the circular arc so that the length of the fence is exactly* $\frac{2\pi}{3}$, *one-third the circumference of the pen*. Why not one-half? As may be evident from Figure 3, there is no value of r which yields a fence of length π.

Before attempting a solution, let us state a version of this problem in higher dimensions. Generalizing to $\mathbb{R}^3$ is straightforward: the "fence" should replace the dark spherical cap in the left side of Figure 2, and the comparison should involve the surface area of the fence and the surface area of the boundary of the pen. However, we will see that there is no value of r for which the surface area of the spherical cap attains even one-third the area of the unit sphere!

In light of these observations, we define the following variation of the n-dimensional grazing goat problem.

Modified grazing goat problem. *A goat is grazing an n-dimensional field bounded by a unit sphere. What is the radius of the fence (spherical cap) whose* $(n-1)$*-volume is equal to exactly* $\frac{1}{n+1}$ *times the* $(n-1)$*-volume of the boundary sphere?*

The choice of the fractions $\frac{1}{n+1}$ is discussed at greater length towards the end of this paper.

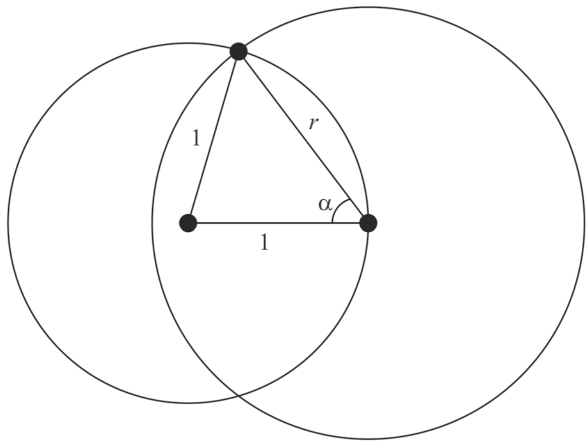

Figure 4 Trigonometric approach to the grazing goat problem.

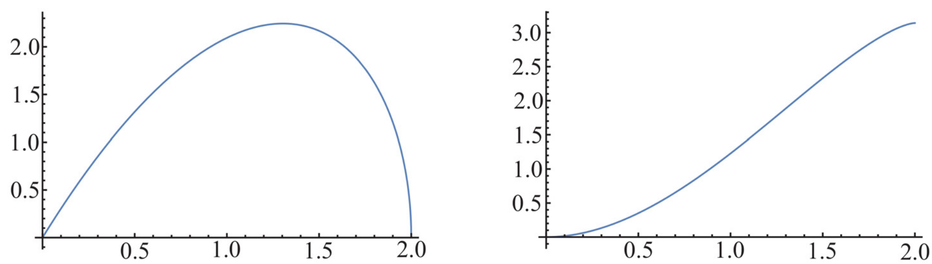

Figure 5 Graphs of $\ell(r)$ (left) and $A(r)$ (right).

Building a fence

We begin our investigation in $\mathbb{R}^2$. Referring to Figure 4, the "fence" is the portion of the larger circle corresponding to the central angle 2α, and hence its length is $2r\alpha$. The law of cosines tells us that $\alpha = \arccos \frac{r}{2}$, which allows us to write the length ℓ of the fence as a function of r:

$$\ell(r) = 2r \cdot \arccos \frac{r}{2}. \tag{4}$$

The graph of $\ell(r)$ is shown on the left side of Figure 5, alongside $A(r)$ (from equation (1)) for comparison. A glance at the graph of $\ell(r)$ confirms that the fence length π is never achieved, and so we seek r such that $\ell(r) = \frac{2\pi}{3}$, one-third the circumference of the pen:

$$3r \cdot \arccos(r/2) = \pi. \tag{5}$$

The solution $r = 1$ is easily verifiable, but approximation techniques are needed to find the second solution $r \approx 1.57285$ (which is not quite $\frac{\pi}{2}$). After examining the solutions to the original n-dimensional grazing goat problem, an integer solution is a refreshing sight, though geometrically it makes sense that an equilateral triangle gives the desired ratio.

We note that the triangle in Figure 4 is the key to solving the original grazing goat problem without the use of calculus. Similarly, one *could* use the integral arc length formula to find $\ell(r)$, but this may be considered a step in the wrong direction – the trigonometric solution is more elegant. This is an important theme of the following section.

Gilding the goat

The modified grazing goat problem in $\mathbb{R}^3$ is perhaps the most interesting among all of the grazing goat problems we know. We hope that this section convinces the reader to share this sentiment.

Glancing back to Figure 2, we seek the radius r such that the surface area of the dark spherical cap is one-fourth the area of the unit sphere. The surface area of the cap may be found by applying the following property of the 2-dimensional sphere, the discovery of which dates back to Archimedes:

The equal area zones property. *If a sphere is sliced by two parallel planes, the surface area of the zone of the sphere lying between the planes is proportional to the distance between the planes, and is therefore independent of their location.*

This property can be checked easily with calculus, and we consider it a wonderful and enlightening exercise for students learning about surfaces of revolution. Archimedes' method may also be used, as Polya suggests [**15**, Ch. XI, Prob. 3]. More recently, the Equal Area Zones property has been studied from a variety of perspectives [**3**], [**5**], [**6**], [**16**], and [**17**].

Archimedes showed that the surface area of a zone is equal to the surface area of the corresponding zone of the circumscribing cylinder (see Figure 6). Explicitly, if the sphere has radius r and the zone has height h, the surface area of the zone is $2\pi rh$.

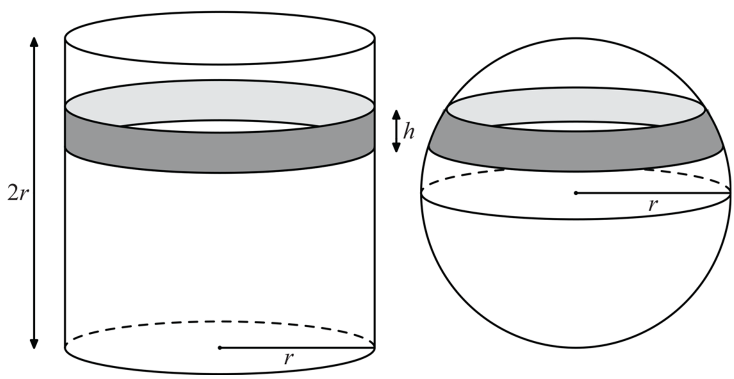

Figure 6 Surface area of cylindrical zone = surface area of spherical zone = $2\pi rh$.

The spherical cap of Figure 2 fits the description of a zone between parallel planes, so we can apply the Equal Area Zones property to find the surface area. In this case, the cap has radius r and height $r - \frac{r^2}{2}$, and so the surface area is given by the formula:

$$S(r) = -\pi r^3 + 2\pi r^2. \tag{6}$$

Setting $S(r) = \pi$ yields a surprising result. We have

$$0 = -r^3 + 2r^2 - 1 = -(r-1)(r^2 - r - 1), \tag{7}$$

for which the two solutions in the interval $[0, 2]$ are $r = 1$ and $r = \varphi = \frac{1+\sqrt{5}}{2} \approx 1.618034$, the golden ratio!

We now have the following justification for our claim that this is the most interesting among grazing goat problems.

- It is the only problem of dimension greater than 2 for which we have a solution without calculus. (Archimedes' result predated modern calculus by nearly two millennia!)
- It is the *only* problem for which all solutions can be reasonably found by hand (ignoring the possibility of using the quartic formula to solve for r_3 in Table 1).
- The solution unifies three special numbers, as stated formally in the proposition below.

Proposition 1. *If a sphere with radius equal to the golden ratio φ is positioned so that its center lies on the surface of a sphere of radius 1, then the surface area of the portion of the golden sphere contained in the unit sphere is precisely π.*

A geometric interpretation of this result would be welcome.

The intersection of two spheres

Proposition 1 is reminiscent of another fascinating property of the 2-sphere, which we feel compelled to share here. Though certainly known to many, we have seen this property only once in the literature, in Polya [**15**, Ch. X, Prob. 39; Ch. XI, Prob. 4]. The property may be considered a consequence of the Equal Area Zones property, and we wonder if it was known to Archimedes.

The sphere intersection property. *Let $2b \geq a > 0$. If a sphere S_a with radius a is positioned so that its center lies on the surface of a sphere S_b of radius b, then the surface area of the portion of S_b contained in S_a is πa^2, independent of the value of b.*

Notice the important distinction from the proposition above: here we study the surface area of the "second" sphere inside the "first," whereas the proposition concerns the surface area of the first sphere inside the second.

Before offering a proof, we examine the two limiting cases. First, when $2b = a$, the entire sphere S_b is contained in S_a, and its surface area is $4\pi \cdot (\frac{a}{2})^2 = \pi a^2$. At the other extreme, we may imagine the "infinite sphere" when $b = \infty$. This sphere arises when we take the left sphere S_b in Figure 7 and increase its radius to infinity while keeping its rightmost point fixed at the center of the right sphere S_a. The resulting "infinite sphere" is a plane passing through the center of S_a. The portion contained in S_a is a disk of radius a, for which the area is πa^2.

For intermediate values of b, we prove the property using Figure 7, which is a cross section of the spherical intersection described above: the sphere S_a is on the right, positioned with its center on the surface of S_b. The portion of S_b lying inside S_a is a spherical cap of height h, so by the equal area zones property, its surface area is $2\pi bh$. Now similarity of the largest and the smallest triangles in Figure 7 yields the equality:

$$\frac{a}{2b} = \frac{h}{a}.$$

Therefore $2bh = a^2$ and so the surface area is $2\pi bh = \pi a^2$, independent of the value of b.

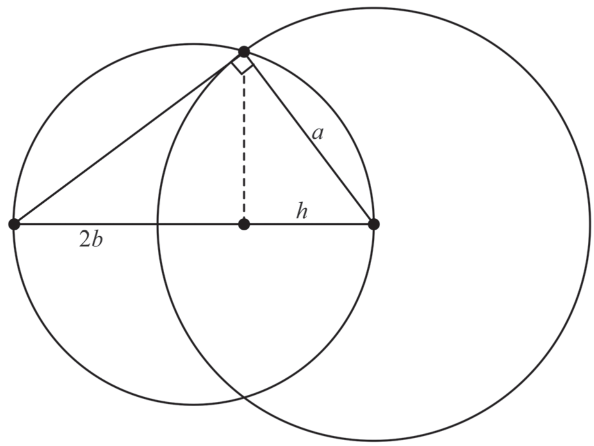

Figure 7 Cross-section of the intersection of the spheres S_a (right) and S_b (left).

Spheres in higher dimensions

The Equal Area Zones Property and the Sphere Intersection Property do not generalize to spheres of higher dimension, but these spheres do exhibit a number of curious properties which we examine in the next section. However, we must first develop some of the language and notation necessary for our higher-dimensional analysis.

The modified grazing goat problem is most conveniently discussed in the language of hypersurfaces of revolution. (The unacquainted reader may find a thorough, accessible introduction to hypersurfaces of revolution in either Coll and Harrison [**4**] or Eisenberg [**8**].) Fortunately, the *idea* of the grazing goat problem is independent of the dimension under investigation, so there is no harm in using the left side of Figure 2 to imagine the problem in any dimension.

To compensate for the change in dimensions, we need only revolve the picture on the right side of Figure 2 through more dimensions to yield two $(n-1)$-dimensional spheres as hypersurfaces of revolution in $\mathbb{R}^n$. We account for this change analytically by replacing the formula for the surface area of a surface of revolution with the corresponding formula for the $(n-1)$-volume of an $(n-1)$-dimensional hypersurface of revolution. This formula will allow us to compute the $(n-1)$-volume of both the spherical cap and the $(n-1)$-sphere.

We use the notation a_k to represent the k-volume of the unit k-dimensional hypersphere. In this notation, $a_1 = 2\pi$ is the circumference of the unit circle and $a_2 = 4\pi$ is the surface area of the unit sphere. Now, if H is an $(n-1)$-dimensional hypersurface of revolution generated by revolving the function $h(x)$, $a \le x \le b$, the $(n-1)$-volume of H is given by

$$\mathrm{Vol}(H) = a_{n-2} \int_a^b h(x)^{n-2} \sqrt{1 + h'(x)^2}\, dx. \tag{8}$$

Notice that when $n = 3$, equation (8) reduces to the formula for the surface area of a surface of revolution in $\mathbb{R}^3$. A formal proof of equation (8) may be found in Aberra and Agrawal [**1**].

The unit $(n-1)$-dimensional sphere can be obtained by revolving the function $h(x) = \sqrt{1-x^2}$, $-1 \le x \le 1$, so we can plug this $h(x)$ into equation (8) to find the $(n-1)$-volume of the $(n-1)$-dimensional sphere. Since

$$h(x) \cdot \sqrt{1 + h'(x)^2} = 1,$$

we obtain the following recursive formula for the volumes of higher-dimensional spheres:

$$a_{n-1} = a_{n-2} \int_{-1}^{1} \left(\sqrt{1-x^2}\right)^{n-3} \, dx. \tag{9}$$

For $n = 3$, this gives the surface area of the unit 2-sphere, assuming we know the circumference of a circle:

$$a_2 = 2\pi \int_{-1}^{1} 1 \, dx = 4\pi.$$

We also compute

$$a_3 = 4\pi \int_{-1}^{1} \sqrt{1-x^2} \, dx = 2\pi^2,$$

$$a_4 = 2\pi^2 \int_{-1}^{1} 1 - x^2 \, dx = \frac{8}{3}\pi^2.$$

The modified grazing goat problem in n dimensions

Returning to our grazing goat problem in dimension n, we seek the $(n-1)$-volume of the fence—which is the portion of the spherical cap of radius r which lies inside the pen. Glancing back to Figure 2, the fence is obtained by revolving the graph of $g(x) = \sqrt{r^2 - x^2}$, $-r \le x \le -\frac{r^2}{2}$, to generate an $(n-1)$-dimensional spherical cap in n-dimensional space. Letting $V_{n-1}(r)$ represent the $(n-1)$-volume of this spherical cap, we have from equation (8):

$$\begin{aligned} V_{n-1}(r) &= a_{n-2} \int_{-r}^{-r^2/2} g(x)^{n-2} \sqrt{1 + g'(x)^2} \, dx \\ &= a_{n-2} \cdot r \int_{-r}^{-r^2/2} \left(\sqrt{r^2 - x^2}\right)^{n-3} \, dx, \end{aligned} \tag{10}$$

where the second equality follows from the definition of $g(x)$. Graphs for select values of n may be seen in Figure 8. Note that $V_1(r) = \ell(r)$ and $V_2(r) = S(r)$, from equations (4) and (6) .

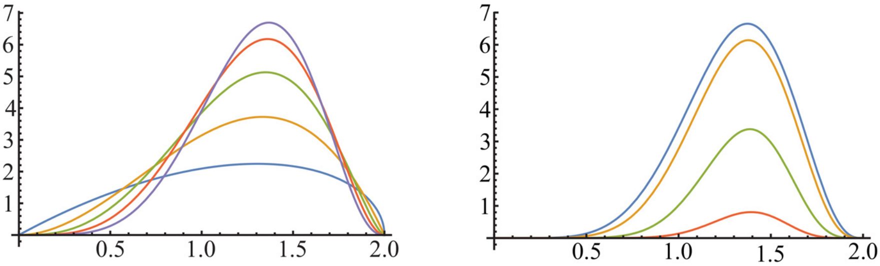

Figure 8 Graphs of select $V_n(r)$. Left: $V_1(r)$ (smallest peak) through $V_5(r)$ (largest peak). Right: $V_6(r)$ (largest peak), $V_7(r)$, $V_{10}(r)$, $V_{14}(r)$ (smallest peak).

In each dimension n, we seek the values of r satisfying the equation $V_{n-1}(r) = \frac{a_{n-1}}{n+1}$. For small values of n, $V_{n-1}(r)$ and a_{n-1} can be computed directly by equations (9) and

n	**Modified grazing goat equation**	**Approx. solutions**
2	$0 = 3r \arccos \frac{r}{2} - \pi$	$1,\ 1.57285$
3	$0 = -r^3 + 2r^2 - 1$	$1,\ \frac{1+\sqrt{5}}{2} \approx 1.61803$
4	$0 = -5r^4\sqrt{4-r^2} + 20r^3 \arccos \frac{r}{2} - 4\pi$	$1.01461,\ 1.63232$
5	$0 = 3r^7 - 36r^5 + 48r^4 - 16$	$1.03061,\ 1.63705$
6	$0 = (7r^8 - 70r^6)\sqrt{4-r^2}$ $+168r^5 \arccos \frac{r}{2} - 24\pi$	$1.04561,\ 1.63789$
11	$0 = -105r^{19} + 2160r^{17} - 18144r^{15}$ $+80640r^{13} - 241920r^{11} + 196608r^{10} - 32768$	$1.10114,\ 1.62761$
21	(omitted)	$1.16026,\ 1.60424$
31	(omitted)	$1.19284,\ 1.58755$

TABLE 2: Solutions to the n-dimensional modified grazing goat problem in select dimensions

(10). We have computed a_3 and a_4 above, and we invite the reader to compute $V_3(r)$ and $V_4(r)$; the answer can be checked against Table 2. As n grows, the computations quickly become unwieldy, and so we have appealed to software to fill the latter half of Table 2. We note that the rows for $n = 2$ and $n = 3$ were found above in (5) and (7).

Intuition in high dimensions

Although we can set up the integral formula in any dimension, it seems unsatisfying that a more compact solution eludes us. This is reminiscent of the original n-dimensional problem [**10**], and in fact, it is an additional reason for our preference of the 3-dimensional case. Nevertheless, we try to overcome this dissatisfication by gaining intuition for both problems in high dimensions. Along the way we introduce the reader to a number of curious properties of higher-dimensional spheres.

The modified grazing goat problem—high-dimensional analysis How can we be sure that the grazing goat equation $V_{n-1}(r) = \frac{a_{n-1}}{n+1}$ has solutions for all n? Two facts about higher-dimensional spheres allow us to justify this claim.

Let v_n represent the n-volume of the n-dimensional solid unit ball. For example, $v_1 = 2$ is the length of the interval $[-1, 1]$, $v_2 = \pi$ is the area enclosed by the unit circle, $v_3 = \frac{4}{3}\pi$ is the volume of the solid ball in $\mathbb{R}^3$. In this notation, a_{n-1} is the $(n-1)$-volume of the boundary of the n-dimensional solid unit disk.

We compare the volume of a disk to the volume of its boundary sphere

$$\frac{v_2}{a_1} = \frac{\pi}{2\pi} = \frac{1}{2} \qquad \text{and} \qquad \frac{v_3}{a_2} = \frac{\frac{4}{3}\pi}{4\pi} = \frac{1}{3}.$$

This pattern continues in all dimensions:

Fact 1. *For* $n \geq 2$, $v_n = \frac{a_{n-1}}{n}$.

We also note that the sequence of volumes v_n, $\{2, \pi, \frac{4}{3}\pi, \dots\}$, and the sequence of volumes a_n, $\{2\pi, 4\pi, 2\pi^2, \frac{8}{3}\pi^2, \dots\}$, both appear to be increasing sequences. It is somewhat surprising to learn that this pattern does *not* continue for all n.

Fact 2. *The volume v_n increases for $1 \leq n \leq 5$ and decreases for $n \geq 5$. The volume a_n increases for $1 \leq n \leq 6$ and decreases for $n \geq 6$. As $n \to \infty$, v_n and a_n both approach* 0.

We are now ready to prove the following.

Proposition 2. *For every $n \geq 2$, the n-dimensional modified grazing goat equation*

$$V_{n-1}(r) = \frac{a_{n-1}}{n+1}$$

admits two solutions.

Proof. We begin with the specific case $r = \sqrt{2}$, so that the graph of $g(x)$ passes through the north pole of the pen, and therefore, so does the spherical cap which represents the fence. The largest cross-section of this spherical cap is the unit $(n-1)$-dimensional disk, which passes through the poles and is parallel to the tangent plane at the leftmost point on the sphere. The spherical cap appears as a slightly bulged copy of this disk, which guarantees that the bulged cap has more volume than its flat counterpart: $V_{n-1}(\sqrt{2}) > v_{n-1}$. Now assume that the ambient dimension $n \geq 6$ and apply the two facts above to the inequality we have just obtained:

$$V_{n-1}(\sqrt{2}) > v_{n-1} > v_n = \frac{a_{n-1}}{n} > \frac{a_{n-1}}{n+1}. \tag{11}$$

For all n, we have

$$V_{n-1}(0) = V_{n-1}(2) = 0,$$

and as long as $n \geq 6$, we have

$$V_{n-1}(\sqrt{2}) > \frac{a_{n-1}}{n+1}.$$

By the intermediate value theorem, there exist two solutions to the modified grazing goat problem: one in the interval $[0, \sqrt{2}]$, and one in the interval $[\sqrt{2}, 2]$. The remaining cases $n < 6$ are handled explicitly in Table 2. ∎

The inequalities in (11) actually guarantee that the fraction $\frac{1}{n}$ is sufficient in higher dimensions, but our earlier discussion showed that it was problematic in dimensions $n = 2$ and $n = 3$.

The following question seems very difficult.

Question 1. *As $n \to \infty$, how do the solutions to the modified grazing goat problem behave?*

In the original goat problem, it is known that the solutions tend to $\sqrt{2}$ as $n \to \infty$. The situation is different in the modified problem since the limit of the solutions is affected by the choice $\frac{1}{n+1}$. If the decay of this sequence is "too fast," the two solutions could approach 0 and 2, respectively. There is some additional discussion on the choice $\frac{1}{n+1}$ in the conclusion of this paper.

The original grazing goat problem—high-dimensional analysis Fraser's original n-dimensional grazing goat problem asks which length of rope r_n is needed to halve the n-volume of the goat pen. When the rope has length $\sqrt{2}$, the fence passes through the north and south poles of the pen, so the entire right half of the pen is grazeable in this case, regardless of the dimension n. Thus, $\sqrt{2}$ is a strict upper bound for all solution lengths.

If the radius is instead $\sqrt{2} - \varepsilon$, then the grazeable region still extends significantly into the left half of the pen, so surely the grazeable volume should still exceed half of the volume of the pen. But this is not the case! This fallacious argument preys on certain false assumptions we tend to make by imagining the setup in 3-dimensional space. The following facts clarify the strange nature of volume accumulation in higher-dimensional balls.

Let $v_n(R)$ be the n-volume of the n-ball of radius R and $a_{n-1}(R)$ be the $(n-1)$-volume of its boundary. In small dimensions, we have

$$a_1(R) = 2\pi R, \qquad a_2(R) = 4\pi R^2, \qquad v_2(R) = \pi R^2, \qquad v_3(R) = \frac{4}{3}\pi R^3.$$

In general, we have:

Fact 3. *For all $n \geq 2$, $v_n(R) = v_n \cdot R^n$ and $a_{n-1}(R) = a_{n-1} \cdot R^{n-1}$.*

Though not relevant to the grazing goat problem, it is interesting to note that combining Facts 1 and 3 yield

$$\frac{\mathrm{d}}{\mathrm{d}R} v_n(R) = \frac{\mathrm{d}}{\mathrm{d}R}(v_n \cdot R^n) = n \cdot v_n \cdot R^{n-1} = a_{n-1} \cdot R^{n-1} = a_{n-1}(R).$$

In some sense, the volume of the boundary sphere is the derivative of the volume of the solid disk. See [7] for some discussion on this property.

Though Fact 3 is not particularly surprising, it yields the surprising consequence that for higher-dimensional balls, almost all of the volume is concentrated near the boundary.

Fact 4. *For any small positive δ, $0 < \delta < 1$, we have*

$$\lim_{n\to\infty} \frac{v_n(1-\delta)}{v_n(1)} = \lim_{n\to\infty} \frac{(1-\delta)^n}{1^n} = 0.$$

Said differently, as $n \to \infty$, almost all of the volume of v_n is contained within distance δ of the boundary sphere! The interested reader may find further exposition in the Vershynin's lecture notes [18] and more advanced, probabilistic-type treatments in Ball [2], Ledoux [12], and Matoušek [13].

These facts lead us to the following intuitive idea which captures the spirit of the convergence $r_n \to \sqrt{2}$. As n gets large, the region near the boundary contains increasingly more volume than the center, and so the problem of halving the volume of the pen becomes more and more equivalent to the problem of enclosing half of the volume of the *boundary* of the pen, which occurs for all n at $r = \sqrt{2}$.

Conclusion

We leave the reader with a few final questions. Let $C_n = \frac{1}{n+1}$, so that the modified grazing goat equation takes the form $V_{n-1}(r) = C_n \cdot a_{n-1}$. The conditions we sought for a suitable sequence of fractions C_n were fairly mild; we only required that the grazing goat equation had solutions in all dimensions. But this is true of many sequences C'_n. What can be said about different sequences C'_n? For example

Question 2. *If C'_n is any sequence, what conditions on C'_n guarantee that solutions of the equation*

$$V_{n-1}(r) = C'_n \cdot a_{n-1}$$

approach $\sqrt{2}$ as $n \to \infty$?

Alternatively, let us explicitly define

$$C'_n := \frac{V_{n-1}(1)}{a_{n-1}}.$$

Geometrically, C'_n represents the ratio:

$$\frac{\text{volume of a cap of } S^{n-1} \text{ of height } 1/2}{\text{volume of } S^{n-1}}.$$

Analytically, C'_n is defined so that in every dimension, $r = 1$ is a solution to

$$V_{n-1}(r) = C'_n \cdot a_{n-1}.$$

For each dimension n, this equation has the second solution in the interval $[\sqrt{2}, 2]$. Geometrically, the second solution $\tilde{r}_n$ would satisfy

$$\text{volume of a cap of } S^{n-1}(\tilde{r}_n) \text{ of height } \frac{2\tilde{r}_n - \tilde{r}_n^2}{2} = \text{volume of a cap of } S^{n-1} \text{ of height } 1/2.$$

Question 3. *From Table 2, $\tilde{r}_2 \approx 1.57285$ and $\tilde{r}_3 = \varphi$. What are the solutions for other n? What is the asymptotic behavior of the sequence $\tilde{r}_n$? What happens if we repeat this setup for the golden ratio φ in place of 1?*

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Summary. We consider a variation of the classical Grazing Goat Problem and we explore connections to a number of surprising properties of spheres, including the curious volume-accumulation in higher-dimensional spheres and the Hat-Box (or Equal Area Zones) Property of the 2-sphere, originally due to Archimedes.

MICHAEL HARRISON (MR Author ID: 1007474) is currently a Member at the Institute for Advanced Study, having completed his Ph.D. at Pennsylvania State University in 2017 under the guidance of Sergei Tabachnikov. He studies various objects in differential topology, especially line fibrations of Euclidean space, billiards, and totally nonparallel immersions. He does not yet own any goats but enjoys goat cheese pizza.

Bounding Monochromatic Triangles Using Squares

RICHARD EHRENBORG
University of Kentucky
Lexington, KY 40506
richard.ehrenborg@uky.edu

A classic exercise is the following: At a party with six people, show that there are either three people who know each other pairwise or three people who do not know each other pairwise. A nice extension is to show that there are at least two such triplets among the six partygoers. A more mathematical way to state this fact is in terms of graph colorings: Color the edges of the complete graph on 6 vertices with the two colors red and blue. Show that there are at least two monochromatic triangles.

Goodman [1] extended this problem to the complete graph on n vertices and provided the best lower bound for the number of monochromatic triangles. We show that we can obtain this lower bound by expressing the problem algebraically and using the basic fact that squares of real expressions are non-negative. This basic fact establishes the lower bound directly in the case $n \equiv 1 \mod 4$. The case when n is even follows by observing that squares of odd integers are at least 1. Finally, we need to tweak the argument to complete the last case when $n \equiv 3 \mod 4$.

Theorem 1 (Goodman). *Let n be at least* 3. *Color the edges of the complete graph K_n with two colors, red and blue. Then the number of monochromatic triangles is at least*

$$\begin{cases} n \cdot (n-1) \cdot (n-5)/24 & \text{if } n \equiv 1 \mod 4, \\ n \cdot (n-2) \cdot (n-4)/24 & \text{if } n \equiv 0 \mod 2, \\ (n+1) \cdot (n-3) \cdot (n-4)/24 & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Furthermore, there are colorings that achieve these bounds.

Proof. Let the vertex set of the graph be $\{1, 2, \ldots, n\}$. Thus, an edge ij in the graph is an unordered pair. To each edge ij, associate two variables x_{ij} and x_{ji}, that we set to be equal. Let $x_{ij} = x_{ji}$ be 1 if the edge ij is red and otherwise let $x_{ij} = x_{ji}$ be -1. Hence, for all edges ij we have $x_{ij}^2 = 1$.

Note that the product $(1 + x_{ij}) \cdot (1 + x_{jk}) \cdot (1 + x_{ki})$ is equal to 8 if $x_{ij} = x_{jk} = x_{ki} = 1$ and zero otherwise. Similarly, the product $(1 - x_{ij}) \cdot (1 - x_{jk}) \cdot (1 - x_{ki})$ is equal to 8 if $x_{ij} = x_{jk} = x_{ki} = -1$ and zero otherwise. Hence their sum is 8 if the triangle ijk is monochromatic and 0 otherwise. Next we have

$$(1 + x_{ij}) \cdot (1 + x_{jk}) \cdot (1 + x_{ki}) + (1 - x_{ij}) \cdot (1 - x_{jk}) \cdot (1 - x_{ki}) = 2 \cdot \bigl(1 + x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij}\bigr)$$

since the monomials of odd degree cancel, and the monomials of even degree double. By dividing the above expression by 8 and summing over $1 \leq i < j < k \leq n$ we obtain that the number of monochromatic triangles p is given by

$$p = \sum_{1 \leq i < j < k \leq n} \frac{1}{4} \cdot (1 + x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij})$$

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MSC: Primary 05D10, Secondary 05C55; 12D15

$$= \frac{1}{4} \cdot \binom{n}{3} + \frac{1}{4} \cdot \sum_{1 \le i < j < k \le n} (x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij}),$$

since the sum contains $\binom{n}{3}$ terms. Observe that the sum is over all paths of length 2, where the first, second and third terms correspond to where the center vertex of the path is the middle, largest, respectively, the smallest indexed vertex. Hence, we can express the sum by first selecting the center of the path and then the remaining two vertices of the path, that is,

$$p = \frac{1}{4} \cdot \binom{n}{3} + \frac{1}{4} \cdot \sum_{j=1}^{n} \sum_{\substack{1 \le i < k \le n \\ i,j,k \text{ distinct}}} x_{ij}x_{jk}$$

$$= \frac{1}{4} \cdot \binom{n}{3} + \frac{1}{8} \cdot \sum_{j=1}^{n} \sum_{\substack{1 \le i < k \le n \\ i,j,k \text{ distinct}}} 2x_{ij}x_{jk}.$$

Now using that $x_{ij} = \pm 1$, we add the zero quantity

$$-n + 1 + \sum_{i=1, i \ne j}^{n} x_{ij}^2 = \sum_{i=1, i \ne j}^{n} (x_{ij}^2 - 1) = 0$$

to the inside of the outer sum.

$$p = \frac{1}{4} \cdot \binom{n}{3} + \frac{1}{8} \cdot \sum_{j=1}^{n} \left(-n + 1 + \sum_{\substack{i=1 \\ i \ne j}}^{n} x_{ij}^2 + \sum_{\substack{1 \le i < k \le n \\ i,j,k \text{ distinct}}} 2x_{ij}x_{jk} \right)$$

$$= \frac{1}{4} \cdot \binom{n}{3} - \frac{n \cdot (n-1)}{8} + \frac{1}{8} \cdot \sum_{j=1}^{n} \left(\sum_{\substack{i=1 \\ i \ne j}}^{n} x_{ij} \right)^2 .$$

Since the sum of squares is non-negative we obtain the bound

$$p \ge \frac{1}{4} \cdot \binom{n}{3} - \frac{n \cdot (n-1)}{8} = \frac{n \cdot (n-1) \cdot (n-5)}{24}.$$

Note that when $n \equiv 1 \bmod 4$ this bound is achieved by taking an Eulerian circuit of K_n and since the number of edges, $\binom{n}{2}$, is even, we can color the edges of the circuit, alternating red and blue. Note that each vertex is adjacent to the same number of red edges as blue edges, yielding that

$$\sum_{i=1, i \ne j}^{n} x_{i,j} = 0.$$

This construction is displayed in Figure 1.

When n is even, note that the sum $\sum_{i=1, i \ne j}^{n} x_{i,j}$ is odd and hence we have the sharper inequality

$$\left(\sum_{i=1, i \ne j}^{n} x_{i,j} \right)^2 \ge 1,$$

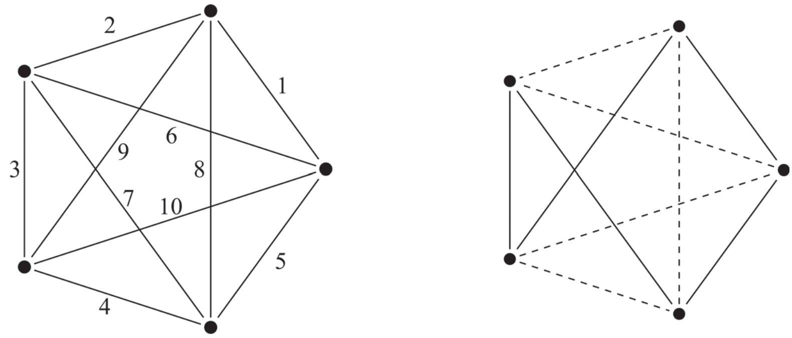

Figure 1 The construction for obtaining the bound for the number of monochromatic triangles for $n \equiv 1 \bmod 4$. Odd labeled edges of the Eulerian circuit are colored red, whereas the even labeled edges are colored blue.

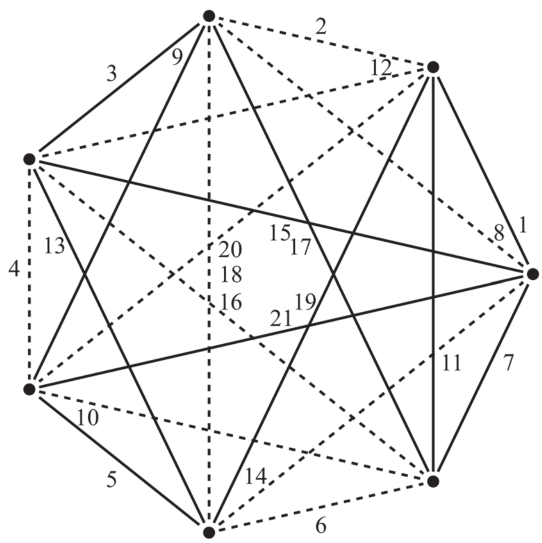

Figure 2 The construction for $n \equiv 3 \bmod 4$. Note that the number of edges is odd. Moreover the most rightmost vertex is incident to the first and the last edge of the Eulerian circuit, and hence this vertex is incident to 2 more red edges than blue.

yielding the stronger bound

$$p \geq \frac{1}{4} \cdot \binom{n}{3} - \frac{n \cdot (n-1)}{8} + \frac{n}{8} = \frac{n \cdot (n-2) \cdot (n-4)}{24}.$$

To attain this lower bound, color the edge ij blue if $i \equiv j \bmod 2$. Each vertex is incident with $n/2 - 1$ blue edges and $n/2$ red, yielding

$$\sum_{i=1, i \neq j}^{n} x_{i,j} = 1,$$

for each vertex j.

The remaining case to study is $n \equiv 3 \bmod 4$. Note that there is no coloring such that each vertex is incident to the same number of red as blue edges, since this would imply that the total number of red edges is the same as the total number of blue edges.

But the total number of edges is $\binom{n}{2}$, which is odd. Thus not all the sums $\sum_{i=1, i\neq j}^{n} x_{i,j}$, which are even, can be zero. At least one has to be non-zero, and smallest even number squared is $(\pm 2)^2 = 4$. Thus the improved lower bound is

$$p \geq \frac{1}{4} \cdot \binom{n}{3} - \frac{n \cdot (n-1)}{8} + \frac{4}{8} = \frac{(n+1) \cdot (n-3) \cdot (n-4)}{24}.$$

Alternatively, note that since $n \equiv 3 \bmod 4$ we have $n \cdot (n-1) \cdot (n-5) \equiv 4 \bmod 8$ so that the previous bound is not an integer. Thus we can add $1/2$ to the bound and still keep it as a lower bound. Again, pick an Eulerian circuit and color the edges, alternating red and blue, starting with red and ending with red. Every vertex but the first vertex in the circuit will be incident to the same number of red and blue edges. The first vertex in the circuit is incident to two more red edges than blue, hence reaching the bound. This construction is shown in Figure 2. ■

In Goodman's original proof, he expressed the number of monochromatic triangles in terms of the degree sequence $(d_0, d_1, \ldots, d_{n-1})$, where d_i is the number of vertices incident with i red edges. Next, he defines two local moves on degree sequences. He shows that these moves decreases his expression and that together they reach the absolute minimum. Our proof is reminiscent of the proof given by Sauvé [**3**]. He associated a weighting with pairs of adjacent edges and obtained a different weighting scheme. For more treatments of sum of squares and their usages, see [**2**] and the references therein.

Acknowledgments The author thanks the referee and the editor for suggestions improving the exposition and Theodore Ehrenborg for reading an earlier draft. This work was also partially supported by a grant from the Simons Foundation (#429370 to Richard Ehrenborg).

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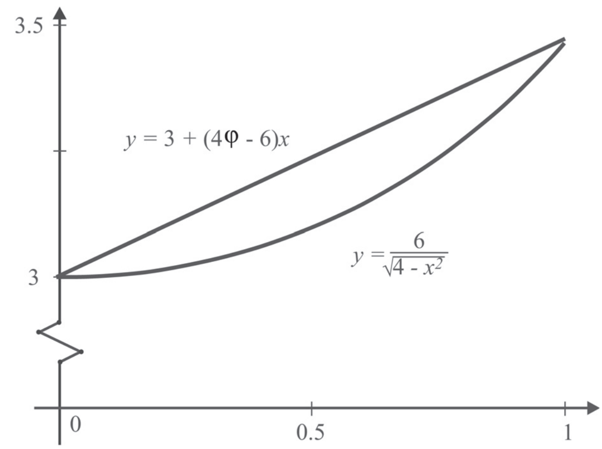
Summary. Goodman gave the lower bound on the number of monochromatic triangles of the complete graph on n vertices when the edges are colored with two colors. We reprove his result by giving an algebraic reformulation and utilizing that squares of real expressions are nonnegative.

RICHARD EHRENBORG (MR Author ID: 312322) received his Ph.D. from the Massachusetts Institute of Technology in 1993. He enjoys exploring the connections between combinatorics and other areas of mathematics.

Pi is Less Than Twice Phi

ROGER B. NELSEN
Lewis & Clark College
Portland, OR 97219
nelsen@lclark.edu

We use calculus and the facts that $\pi = 6\arcsin(1/2)$ and the golden ratio $\phi = (1+\sqrt{5})/2$ to answer the question in the title of the article by Morales, Pak, and Panova [1]. That is, we show that $\pi < 2\phi$.



$$\pi = \int_0^1 \frac{6}{\sqrt{4-x^2}}\,dx < \int_0^1 3 + (4\phi - 6)x\,dx = 2\phi.$$

At the first glance, it may appear that the line segment in the figure is a secant joining the endpoints of the curve. But that is not the case; the ordinate of the right endpoint of the line segment is $2\sqrt{5}-1 \approx 3.4721$ while the ordinate of the right endpoint of the curve is $2\sqrt{3} \approx 3.4641$. Replacing the line segment with the secant yields a trapezoidal rule approximation to the integral for π and a slightly sharper (but perhaps less interesting) inequality: $\pi < 1.5 + \sqrt{3}$.

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Summary. We use integral calculus to establish the inequality that π is less than 2ϕ.

ROGER B. NELSEN (MR Author ID: 237909) is a professor emeritus at Lewis & Clark College, where he taught mathematics and statistics for 40 years.

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MSC: 51M16

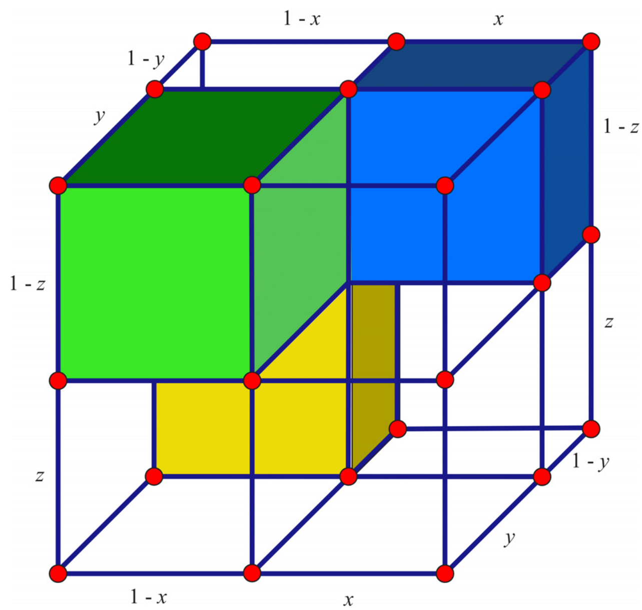
An Algebraic Inequality

WEI-DONG JIANG
Weihai Vocational College
Weihai 264210, ShanDong province, China
jackjwd@163.com

Let x, y, z be positive real numbers, $0 < x, y, z < 1$. Prove

$$x(1-y)(1-z) + (1-x)y(1-z) + (1-x)(1-y)z < 1 \tag{1}$$

This appeared as a problem in the Chinese publication *Mathematics Teaching* [1]. We present a visual proof of this result.



Note that the cube has side length 1, and therefore volume 1. Equation (1) is now evident.

Since the cube is divided into eight small cuboids, we have the following identity:

$$x(1-y)(1-z) + (1-x)y(1-z) + (1-x)(1-y)z + xy(1-z) \\ + yz(1-x) + zx(1-y) + xyz + (1-x)(1-y)(1-z) = 1. \tag{2}$$

From (2), one can obtain several inequalities stronger than (1). For example, the following inequalities hold:

$$x(1-y)(1-z) + (1-x)y(1-z) + (1-x)(1-y)z \\ + xy(1-z) + yz(1-x) + zx(1-y) + xyz < 1.$$

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MSC: 26D15

and

$$x(1-y)(1-z)+(1-x)y(1-z)+(1-x)(1-y)z \\ +xy(1-z)+yz(1-x)+zx(1-y)+(1-x)(1-y)(1-z)<1.$$

Acknowledgments The author appreciates the editor and anonymous referees for their careful corrections and valuable suggestions to the original version of this paper.

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Summary. Construct a cube with edge length 1. By dividing the cube into 8 smaller regions, we give a proof of an algebraic inequality.

WEI-DONG JIANG (MR Author ID: 797990) is a full professor at WeiHai Vocational College. His research interests include classical analysis, special functions, mathematical inequalities, and mathematical means.

PROBLEMS

Proposals

To be considered for publication, solutions should be received by May 1, 2022.

2131. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Recall that a symmedian is the reflection of a median through a vertex across the angle bisector passing through that vertex. The three symmedians of a triangle meet in a point known as the symmedian (or Lemoine or Grebe) point. Let ABC be a triangle with symmedian point S. Let X, Y, and Z be points lying on segments SA, SB, and SC, respectively, such that $\angle XBA \cong \angle YAB$ and $\angle XCA \cong \angle ZAC$. Prove that $\angle ZBC \cong \angle YCB$.

2132. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

A regular tetrahedral die with sides of length 1 is tossed onto a floor having a family of parallel lines spaced 1 unit apart. What is the probability that the die lands on a line?

2133. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

Evaluate the infinite sum

$$\sum_{k=1}^{\infty} 2^{-k} \tan\left(2^{-k}\right).$$

2134. *Proposed by Antonio Garcia, Strasbourg, France.*

Let $N \in M_n(\mathbb{R})$ be a nilpotent matrix. In what follows, $X \in M_n(\mathbb{R})$.

(a) Show that there is always an X such that $N = X^2 + X - I$.
(b) Show that if n is odd, there is no X such that $N = X^2 + X + I$.

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

(c) Show that if $n = 2$ and $N \neq 0$, there is no X such that $N = X^2 + X + I$.

(d) Give examples, when $n = 4$, of an $N \neq 0$ and an X such that $N = X^2 + X + I$ and of an N with no X such that $N = X^2 + X + I$.

2135. *Proposed by Băetu Ioan, "Mihai Eminescu" National College, Botoşani, Romania.*

For $k \in \mathbb{Z}^+$, let $a_n(k)$ denote the number of elements $\sigma \in S_n$, the group of all permutations on an n-element set, such that $\sigma^k = e$, the identity element. We take $a_0(k) = 1$ by convention. Find a closed form for the exponential generating function

$$f_k(x) = \sum_{n=0}^{\infty} \frac{a_n(k)x^n}{n!}.$$

Quickies

1115. *Proposed by Ben Fusaro and John Loase, Fordham University, Bronx, NY.*

Let $\{(x_i, y_i)\}$, $i = 1, \ldots, n$, be a set of data points that do not all lie on a horizontal or on a vertical line. When is the least-square regression line considering x to be an independent variable and y to be dependent the same as that with y independent and x dependent?

1116. *Proposed by Lokman Gökçe, Istanbul, Turkey.*

For which prime numbers p is

$$p^4 - 35p^3 + 365p^2 - 1225p + 1259$$

a prime number?

Solutions

A Maclaurin series with integer coefficients — December 2020

2106. *Proposed by Timothy Hall, PQI Consulting, Cambridge, MA.*

For a fixed positive integer $n > 1$, characterize those integers k such that the Maclaurin series for $f_{n,k}(x) = \sqrt[n]{1 + kx}$ (as a function of x) has integer coefficients.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Let $n = \prod_{i=1}^{\ell} p_i^{\beta_i}$ be the prime factorization of n. We will prove that the Maclaurin series of $f_{n,k}$ has integer coefficients if and only if k is a multiple of $\lambda_n = \prod_{i=1}^{\ell} p_i^{\beta_i+1}$.

Recall that

$$f_{n,k}(x) = \sqrt[n]{1 + kx} = 1 + \sum_{m=1}^{\infty} \binom{1/n}{m} k^m x^m$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^{m-1} \frac{k^m}{n^m\, m!} \prod_{j=1}^{m-1} (jn - 1) x^m$$

Let

$$a_{n,k}(m) = \frac{k^m}{n^m\, m!} \prod_{j=1}^{m-1} (jn-1).$$

As usual, for an integer $m \neq 0$ and a prime p, the largest exponent α such that p^α divides m, will be denoted by $\nu_p(m)$. Note that $a_{n,k}(m) \in \mathbb{Z}$ if and only if

$$\nu_p\left(k^m \prod_{j=1}^{m-1}(jn-1)\right) \geq \nu_p\left(n^m m!\right) \tag{1}$$

for all primes p. Also, if $\nu_p(k) \geq \nu_p(n) + 1$ for all primes p dividing n, then k is a multiple of λ_n.

We recall the well-known fact that

$$\nu_p(m!) = \sum_{r\geq 1} \left\lfloor \frac{m}{p^r} \right\rfloor.$$

In particular, this implies that $\nu_p(m!) \leq m/(p-1) \leq m$.

Necessity. Suppose $a_{n,k}(m) \in \mathbb{Z}$ for all $m \geq 0$. Then (1) holds for all primes p and all $m \geq 0$. Take $m = n$. For p dividing n, we have $\nu_p(jn-1) = 0$. It follows that

$$\nu_p\left(k^n \prod_{j=1}^{n-1}(jn-1)\right) = n\nu_p(k).$$

On the other hand,

$$\nu_p\left(n^n n!\right) \geq (n+1)\nu_p(n) + \nu_p\left((n-1)!\right) \geq (n+1)\nu_p(n) > n\nu_p(n).$$

Thus

$$n\nu_p(k) = \nu_p\left(k^n \prod_{j=1}^{n-1}(jn-1)\right) \geq \nu_p\left(n^n n!\right) > n\nu_p(n).$$

Hence $\nu_p(k) > \nu_p(n)$ or equivalently $\nu_p(k) \geq \nu_p(n) + 1$. Thus k is a multiple of λ_n.

Sufficiency. Suppose that $k = \lambda_n$, and p is an arbitrary prime.

If p divides n, then

$$\nu_p(n^m m!) = m\nu_p(n) + \nu_p(m!) \leq m\nu_p(n) + m = m\nu_p(\lambda_n) = \nu_p(\lambda_n^m).$$

Since $\nu_p(jn-1) = 0$, (1) holds.

If p does not divide n (and hence does not divide λ_n), then

$$\nu_p(n^m m!) = \nu_p(m!) = \sum_{r\geq 1} \left\lfloor \frac{m}{p^r} \right\rfloor.$$

On the other hand, since $\gcd(n, p^r) = 1$, there is a unique j modulo p^r such that $nj \equiv 1 \pmod{p^r}$. It follows that

$$\nu_p\left(\prod_{j=1}^{m-1}(nj-1)\right) = \sum_{r\geq 1} |\{j \in \{1, 2, \ldots, m-1\} : nj \equiv 1 \pmod{p^r}\}|$$

$$\geq \sum_{r\geq 1}\left\lfloor \frac{m}{p^r}\right\rfloor = \nu_p(m!).$$

Since $\nu_p(n) = \nu_p(\lambda_n) = 0$, (1) holds and $a_{n,\lambda_n}(m) \in \mathbb{Z}$ for all $m \geq 0$.

Noting that $a_{n,qu}(m) = q^m a_{n,u}(m)$ shows that if k is a multiple of λ_n, then $a_{n,k}(m) \in \mathbb{Z}$ for all $m \geq 0$, and the claim has been proven.

Also solved by Matthew Creek, Dmitry Fleischman, Northwestern University Math Problem Solving Group, Francisco Perdomo & Ángel Plaza (Spain), Randy K. Schwartz, Albert Stadler (Switzerland), and the proposer. There were three incomplete or incorrect solutions.

Evaluate the improper integral **December 2020**

2107. *Proposed by Seán M. Stewart, Bomaderry, Australia.*

Evaluate

$$\int_0^{\infty} \frac{\log\left(1+x^6\right)}{1+x^2}\, dx.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.
We need the following result. Let

$$J = \int_0^{\pi/2} \ln(\cos\theta)\, d\theta = \int_0^{\pi/2} \ln(\sin\theta)\, d\theta.$$

Then

$$J + J = \int_0^{\pi/2} \ln(\sin\theta\cos\theta)\, d\theta = -\frac{1}{2}\ln\left(\frac{\pi}{2}\right) + \int_0^{\pi/2} \ln(\sin(2\theta))\, d\theta$$

$$= -\frac{1}{2}\ln\left(\frac{\pi}{2}\right) + \frac{1}{2}\int_0^{\pi} \ln(\sin\theta)\, d\theta = -\frac{1}{2}\ln\left(\frac{\pi}{2}\right) + J,$$

so $J = -\ln(\pi/2)/2$.

With the substitution $x = \tan\theta$,

$$\int_0^{\infty} \frac{\ln(1+x^6)}{1+x^2}\, dx = \int_0^{\pi/2} \ln(1+\tan^6\theta)\, d\theta$$

$$= \int_0^{\pi/2} \ln(1+\tan^2\theta)\, d\theta + \int_0^{\pi/2} \ln(1-\tan^2\theta+\tan^4\theta)\, d\theta$$

$$= -2\int_0^{\pi/2} \ln(\cos\theta)\, d\theta - 4\int_0^{\pi/2} \ln(\cos\theta)\, d\theta +$$

$$\int_0^{\pi/2} \ln(\cos^4\theta - \sin^2\theta\cos^2\theta + \sin^4\theta)\, d\theta$$

$$
\begin{aligned}
&= -6\left(-\frac{\pi}{2}\ln 2\right) + \int_0^{\pi/2} \ln(1 - 3\sin^2\theta\cos^2\theta)\,d\theta \\
&= 3\pi\ln 2 + \int_0^{\pi/2} \ln\left(1 - \frac{3}{4}\sin^2 2\theta\right)\,d\theta \\
&= 3\pi\ln 2 + \int_0^{\pi/2} \ln\left(\frac{1}{4} + \frac{3}{4}\cos^2\theta\right)\,d\theta \\
&= 3\pi\ln 2 + \frac{\pi}{2}\ln\frac{3}{4} + \int_0^{\pi/2} \ln\left(\frac{1}{3} + \cos^2\theta\right)\,d\theta
\end{aligned}
$$

Let

$$I(\lambda) = \int_0^{\pi/2} \ln(\lambda^2 + \cos^2\theta)\,d\theta.$$

Then

$$
\begin{aligned}
I'(\lambda) &= \int_0^{\pi/2} \frac{2\lambda}{\lambda^2 + \cos^2\theta}\,d\theta = \int_0^{\pi/2} \frac{2\lambda\sec^2\theta}{\lambda^2\sec^2\theta + 1}\,d\theta \\
&= \int_0^{\pi/2} \frac{2\lambda\sec^2\theta}{(1+\lambda^2) + \lambda^2\tan^2\theta}\,d\theta = \int_0^{\infty} \frac{2\lambda}{(1+\lambda^2) + \lambda^2 u^2}\,du \\
&= \frac{2}{\sqrt{1+\lambda^2}}\tan^{-1}\frac{\lambda u}{\sqrt{1+\lambda^2}}\bigg|_0^{\infty} = \frac{\pi}{\sqrt{1+\lambda^2}}.
\end{aligned}
$$

Integrating with respect to λ yields $I(\lambda) = \pi\ln(\lambda + \sqrt{1+\lambda^2}) + C$. With

$$I(0) = \int_0^{\pi/2} \ln(\cos^2\theta)\,d\theta = 2\int_0^{\pi/2} \ln(\cos\theta)\,d\theta = 2\left(-\frac{\pi}{2}\ln 2\right) = -\pi\ln 2,$$

it follows that $C = -\pi\ln 2$. Thus,

$$\int_0^{\pi/2} \ln\left(\frac{1}{3} + \cos^2\theta\right)\,d\theta = I\left(\sqrt{\frac{1}{3}}\right) = \pi\ln\left(\sqrt{\frac{1}{3}} + \sqrt{\frac{4}{3}}\right) - \pi\ln 2 = \frac{\pi}{2}\ln 3 - \pi\ln 2,$$

and

$$\int_0^{\infty} \frac{\ln(1+x^6)}{1+x^2}\,dx = 3\pi\ln 2 + \frac{\pi}{2}\ln\frac{3}{4} + \frac{\pi}{2}\ln 3 - \pi\ln 2 = \pi\ln 6.$$

Also solved by Farrukh Ataev (Uzbekistan), Michel Bataille (France), Necdet Batir (Turkey), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Khristo N. Boyadzhiev, Paul Bracken, Hongwei Chen, Bruce E. Davis, Dmitry Fleischman, Subhankar Gayen (India), Russell Gordon, Lixing Han, Shing Hin Jimmy Pa (Canada), John Heuver (Afghanistan), Eugene Herman, Finbarr Holland (Ireland), Walther Janous (Austria), Warren P. Johnson, John Kampmeyer, Omran Kouba (Syria), Kee-Wai Lau (Hong Kong), Omarjee Moubinool (France), Northwestern University Math Problem Solving Group, Volkhard Schindler (Germany), Caleb Soto, Albert Stadler (Switzerland), and the proposer. There was one incomplete or incorrect solution.

Groups with all proper subgroups cyclic **December 2020**

2108. *Proposed by Souvik Dey (graduate student), University of Kansas, Lawrence, KS.*

Let G be a torsion-free abelian group such that every proper subgroup of G is cyclic. Show that G is cyclic.

Solution by Edgar Enochs, University of Kentucky, Lexington, KY, and David Stone, Georgia Southern University, Statesboro, GA.
If G is the trivial group, the claim is certainly true, so we will assume G is non-trivial.

Suppose there exists a non-zero integer n such that the subgroup nG does not equal G. Then nG is cyclic by hypothesis. Consider the map $x \mapsto nx$ from G to itself. It is a homomorphism since G is abelian, and the kernel is trivial since G is torsion-free. Therefore $G \cong nG$ is cyclic.

On the other hand, suppose no such n exists; that is, $nG = G$ for all non-zero integers n. Therefore, G is divisible. Because G is torsion-free and divisible, it is a vector space over the field of rational numbers. Hence it is isomorphic to a direct sum of copies of $\mathbb{Q}$. But $\mathbb{Q}$, hence G, has proper non-cyclic subgroups. For example, the dyadic rationals, $\{m/2^n \mid m, n \in \mathbb{Z}\}$ are such a subgroup. Therefore this case cannot occur and G must be cyclic.

Editor's Note. There are examples of non-abelian torsion-free groups with all proper subgroups cyclic ("Tarski monsters").

Also solved by Anthony Bevelacqua, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia) Paul Budney, Ángel Plaza & Francisco Perdomo (Spain), Michael Reid, Celia Schacht, John H. Smith, Albert Stadler (Switzerland), Edward White & Roberta White, and the proposer. There were three incomplete or incorrect solutions.

Power-preserving continuous maps **December 2020**

2109. *Proposed by George Stoica, Saint John, NB, Canada.*

Let S^1 denote the multiplicative group of complex numbers of absolute value 1. Characterize all continuous maps $f : S^1 \to S^1$ such that $f(z^m) = f(z)^m$ for all $z \in S^1$.

Solution by Francisco Perdomo and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.
Suppose $m \geq 2$, and let $g(z) = f(z)/f(1)$. Then

$$g\left(z^m\right) = \frac{f\left(z^m\right)}{f(1)} = \left(\frac{f\left(z\right)}{f(1)}\right)^m = g\left(z\right)^m$$

with $g(1) = 1$. Let d be the winding number of g. Put $h(z) = g(z)/z^d$. Then $h(1) = 1$, $h\left(z^m\right) = h\left(z\right)^m$, and the winding number of h is 0.

Put $z = e^{i\theta}$, and $h(z) = e^{i\lambda(\theta)}$ with λ a continuous function $\lambda : [0, 2\pi] \longrightarrow \mathbb{R}$, such that $\lambda(0) = 0$ (since $h(1) = 1$) and $\lambda(2\pi) = \lambda(0) = 0$ (since the winding number of $h(z)$ is 0). Now λ can be extended to a continuous periodic function $\kappa : \mathbb{R} \longrightarrow \mathbb{R}$ by $\kappa(\theta) = \lambda(\theta \bmod 2\pi)$. Consider the zeros of κ. Since for all $r \geq 0$

$$h\left(z^{m^r}\right) = h\left(z\right)^{m^r},$$

we have $\kappa\left(m^r\theta\right) \equiv m^r\kappa(\theta) \pmod{2\pi}$. Therefore, if $\theta = 2\pi\ell/m^r$, then $\kappa(\theta) \equiv 0 \pmod{2\pi}$. Hence

$$\kappa\left(\left\{\frac{2\pi\ell}{m^r}\middle|\, \ell \in \mathbb{Z}, r \in \mathbb{N}\right\}\right) \subseteq \{2\pi n | n \in \mathbb{Z}\}.$$

Since κ is continuous, and the image of a dense set under it is contained in a discrete set, it must be constant. Because $\kappa(0) = 0$, $\kappa(\theta) = 0$ for all θ. Consequently, $h(z) = 1$ for all z, $g(z) = z^d$, and $f(z) = f(1)z^d$. Moreover, $f(1) = f(1)^m$, so $f(1)$ is an $(m-1)$st root of unity. Therefore $f(z) = \omega z^d$, where $\omega^{m-1} = 1$.

Conversely, every function of this type satisfies the hypothesis, and the problem is solved.

Editor's Note. The statement of the problem is unclear. The author's intention was for m to be fixed, but the problem could be interpreted as asking for the condition to hold for all m. In the latter case, the solution is $f(z) = z^d$. Correct solutions to either interpretation were credited. The ambiguity is due to the editors, who regret the oversight.

Also solved by Anthony Bevalacqua, Eagle Problem Solvers, Winston Gee, Albert Stadler (Switzerland), and the proposer.

The Elkies point of an isosceles triangle **December 2020**

2110. *Proposed by Rob Downes, Newark Academy, Livingston, NJ.*

Let $\triangle ABC$ be an isosceles triangle with $AB = AC = a$ and $BC = b$. Let X be a point in the interior of $\triangle ABC$ such that the inradii of $\triangle AXB$, $\triangle BXC$, and $\triangle CXA$ are all equal. Express that common radius r in terms of a and b.

Solution by Randy K. Schwartz, Ann Arbor, MI.
Editor's Note. This problem was inspired by Problem 1238 in this journal, proposed by Clark Kimberling (April, 1986) with a solution by Noam Elkies (April, 1987).

It is well known that a triangle with sides of length u, v, and w has an inradius of

$$r = \frac{1}{2}\sqrt{\frac{(-u+v+w)(u-v+w)(u+v-w)}{u+v+w}}. \tag{1}$$

Elkies showed that for an arbitrary triangle ABC, the point X is unique. However, in our case, the intermediate value theorem yields a solution with X lying on the perpendicular bisector of $\overline{BC}$. Therefore, we must have $BX = CX$. (Note: This does not follow solely from the fact that $AB = AC$ and triangles ABX and ACX have the same inradii, as $AB = AC = AX = BX = 1$ and $CX = (3+\sqrt{33})/6$ shows.) Let $x = BX = CX$, $y = AX$, $z = \sqrt{x^2 - b^2/4}$ be the height of $\triangle BXC$, and $h = y + z = \sqrt{a^2 - b^2/4}$ be the height of $\triangle ABC$.

Applying (1) to $\triangle BXC$ yields

$$r = \frac{1}{2}\sqrt{\frac{b^2(2x-b)}{2x+b}}.$$

Solving for x gives

$$x = \frac{b(b^2+4r^2)}{2(b^2-4r^2)}, \quad \text{so } x + \frac{b}{2} = \frac{b^3}{b^2-4r^2}, \quad \text{and } x - \frac{b}{2} = \frac{4br^2}{b^2-4r^2}.$$

Therefore,

$$z = \sqrt{\left(x+\frac{b}{2}\right)\left(x-\frac{b}{2}\right)} = \sqrt{\frac{b^3}{b^2-4r^2}\cdot\frac{4br^2}{b^2-4r^2}} = \frac{2b^2r}{b^2-4r^2}.$$

Now

$$a + x + y = a + x + h - z = a + \frac{b(b^2 + 4r^2)}{2(b^2 - 4r^2)} + h - \frac{2b^2 r}{b^2 - 4r^2}$$
$$= \frac{b\left(a + h + \frac{b}{2}\right) + 2r\left(a + h - \frac{b}{2}\right)}{b + 2r}.$$

Similarly,

$$a + x - y = \frac{b\left(a - h + \frac{b}{2}\right) + 2r\left(-a + h + \frac{b}{2}\right)}{b - 2r}.$$

Letting

$$p = a + \frac{b}{2} + h, \overline{p} = a + \frac{b}{2} - h, q = -a + \frac{b}{2} + h, \quad \text{and } \overline{q} = a - \frac{b}{2} + h,$$

we have

$$a + x + y = \frac{bp + 2r\overline{q}}{b + 2r}, \quad \text{and } a + x - y = \frac{b\overline{p} + 2rq}{b - 2r}.$$

Analogous arguments give

$$a - x + y = \frac{b\overline{q} - 2rp}{b - 2r}, \quad \text{and } \; -a + x + y = \frac{bq - 2r\overline{p}}{b + 2r}.$$

Note that $pq = \overline{pq} = bh$. As a result,

$$b\overline{p} + 2rq = \frac{q}{\overline{q}}(bp + 2r\overline{q}) \text{ and } b\overline{q} - 2rp = \frac{p}{\overline{p}}(bq - 2r\overline{p}).$$

Applying (1) to $\triangle ABX$, we have

$$r = \frac{1}{2}\sqrt{\frac{(-a + x + y)(a - x + y)(a + x - y)}{a + x + y}}$$
$$= \frac{1}{2}\sqrt{\frac{(bq - 2r\overline{p})(b\overline{q} - 2rp)(b\overline{p} + 2rq)}{(bp + 2r\overline{q})(b - 2r)^2}} = \frac{1}{2}\sqrt{\frac{pq(bq - 2r\overline{p})^2}{\overline{pq}(b - 2r)^2}} = \frac{bq - 2r\overline{p}}{2(b - 2r)}.$$

Therefore, r satisfies the quadratic equation

$$4r^2 - 2(b + \overline{p})r + bq = 0, \quad \text{so } r = \frac{b + \overline{p} \pm \sqrt{(b + \overline{p})^2 - 4bq}}{4}.$$

Using $\overline{p} = a + b/2 - h$, $q = -a + b/2 + h$, and $h^2 = a^2 - b^2/4$, we find that $(b + \overline{p})^2 - 4bq = (2a + 7b)(a - h)$. This gives

$$r = \frac{2a + 3b - 2h - 2\sqrt{(2a + 7b)(a - h)}}{8},$$

which can be rewritten as

$$r = \frac{2a + 3b - \sqrt{(2a + b)(2a - b)} - \sqrt{(2a + 7b)(2a + b)} + \sqrt{(2a + 7b)(2a - b)}}{8}.$$

We note that the second root is extraneous, because it gives an inradius greater than $b/2$, which is impossible.

Also solved by Volkhard Schindler (Germany), Albert Stadler (Switzerland), and the proposer. There was one incomplete or incorrect solution.

Answers

Solutions to the Quickies from page 391.

A1115. We claim that this occurs if and only if the (x_i, y_i) are collinear.

If the equations of the least-square regression lines are $y = mx + b$ and $x = \overline{m}y + \overline{b}$, then

$$m = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} \quad \text{and} \quad \overline{m} = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n y_i\right)^2}$$

Without loss of generality, we may assume that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0$. If the two lines are identical, then $\overline{m} = 1/m$. Therefore,

$$\frac{n\sum_{i=1}^n x_i y_i}{n\sum_{i=1}^n x_i^2} = \frac{n\sum_{i=1}^n y_i^2}{n\sum_{i=1}^n x_i y_i},$$

implying that

$$\left(\sum_{i=1}^n x_i y_i\right)^2 = \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right).$$

By the Cauchy–Schwartz inequality, the equality above holds if and only if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are linearly dependent. Since neither of these are the zero vector, the data points lie on the line $y = \lambda x$, $\lambda \neq 0$.

On the other hand, if $y_i = \lambda x_i$, then we have $\overline{m} = 1/m$ and $\overline{b} = b = 0$, so the two lines are identical.

A1116. Let $f(p) = p^4 - 35p^3 + 365p^2 - 1225p + 1259$. For $p = 5$, $f(5) = 509$ is a prime number. We may therefore assume that $p \neq 5$. By Fermat's theorem, $p^4 \equiv 1 \pmod 5$. Therefore $f(p) \equiv 1 + 1259 \equiv 0 \pmod 5$. Hence $5 \mid f(p)$. Since $f(p)$ is a prime number, we must have $f(p) = 5$ and $p^4 - 35p^3 + 365p^2 - 1225p + 1259 = 5$. Now $p^4 - 35p^3 + 365p^2 - 1225p + 1254 = (p-2)(p-3)(p-11)(p-19)$, so the solutions are $p = 2, 3, 5, 11, 19$.

Note: Allowing negative primes does not increase the number of solutions. We have $f(-5) = 137 \cdot 157$, which is not prime. If $p \not\equiv 0 \pmod 5$, the argument above forces $f(p) = -5$, but the rational root test shows that this has no integer solutions.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Stewart, Ian, *What's the Use? How Mathematics Shapes Everyday Life*, Basic Books, 2021; 309 pp, $28. ISBN 978-1-5416-9948-9.

"What is mathematics for? ... [W]e've outsourced most of the mathematics to electronic devices with built-in algorithms.... The elephant isn't even in the room.... Without mathematics, today's world would fall apart. ... I'm going to show you applications to politics, the law, kidney transplants, supermarket delivery schedules, Internet security, movie special effects, and making springs." Stewart concentrates on "unreasonable" applications of mathematics, ones in which there is no connection between the original motivation for the mathematics and the application. He makes a convincing argument that "mathematics is *essential* to today's way of life," and that people should appreciate that fact.

Du Sautoy, Marcus, *Thinking Better: The Art of the Shortcut in Math and Life*, Basic Books, 2021; 336 pp, $28(P). ISBN 978-1-5416-0036-2.

"It was the lure of the shortcut that made me want to become a mathematician." Author Sautoy goes on to conceptualize mathematics as "a celebration of the shortcut." Each chapter begins with a puzzle and concludes with a "pit stop" of practical application. Sautoy illustrates shortcuts achieved by discerning patterns, calculating in binary, changing numbers into geometry, seeking geodesics, making diagrams, minimizing via calculus, employing data, "turning uncertainty into numbers," and systematizing heuristics. The chapters range all over mathematics: summing sequences, solving polynomial equations, winning at games, trigonometry, cycloids, probability, networks, primality testing, and NP problems. This is a fascinating book for the general reader, and its characterization of mathematics as a kind of "better thinking" offers apologists for mathematics a welcome new slogan.

Rising, Gerald R., James R. Matthews, Eileen Schoaff, and Judith Matthews, *About Mathematics*, Linus Learning, 2021; xiv + 300 pp, $32(P). ISBN 978-1-60797-892-3.

The authors emphasize that this innovative book, which is eminently suitable for a liberal arts mathematics course, is a "humanities text, not a mathematics text ... its focus is on appreciation ... rather than the accumulation of technique." Its goals are to make the student comfortable with mathematics and familiar with interesting problems that mathematicians have solved, while offering background for economic decisions and providing enjoyable experiences with mathematics. The last goal is implemented in a novel way that should both entertain and draw in students—through the well-integrated use of what the authors term "panels": QR codes embedded in the text that lead to web applications for a student to explore and experiment with on a cellphone. The panels are key to the approach of the book, which above all is to have students experiment and try things out. No prerequisites are mentioned explicitly, but the text relies on basic familiarity with algebra, geometry, and trigonometry. The chapters cover algorithms; calculating using dimensional units; loans, credit cards, and gambling; ciphers and modular arithmetic; calculus; probability, expectation, and simulation; algorithms for arithmetic; logarithms and exponentials; and alternative geometries. Exercises are provided, together with their answers.

Math. Mag. **94** (2021) 399–400. doi:10.1080/0025570X.2021.1996847

Saracco, Alberto, Mathematics in Disney comics, in *Imagine Math 7: Between Culture and Mathematics*, edited by Michele Emmer and Marco Abate, 189–210; New York: Springer. https://www.researchgate.net/profile/Alberto-Saracco/publication/335566062_Mathematics_in_Disney_Comics/links/5d6d3fd292851c853887d16d/Mathematics-in-Disney-Comics.pdf.

________, Of ducks and paths, of math and mice—the mathematics of Disney comics, video (1:33:29), https://www.youtube.com/watch?v=qE91YzBgWcw.

You may remember the video "Donald in Mathmagic Land" (1959), available now on the internet. Donald's interest in mathematics is being carried on by him, his nephews, and other Disney characters in current comic strips. What ideas about mathematics do those comics convey? Previously, mathematical tasks or notation were used to scare the reader or to illustrate mumbo-jumbo. These days, stories retell versions of classics (grains on a chessboard, Dido's use of a hide to enclose a large area), and those by former engineer Don Rosa feature mathematical notation and the Hindu legacy of a symbol for zero. A few comics have been co-written by mathematicians to teach or popularize mathematics; you can guess what the bridges of Quackenberg in the duck universe corresponds to—and the comic in question even includes a proof! (A surprise to me: Most contemporary Disney comics are produced in Italy and translated.)

Nyquist, Eric, Mathematicians solve decades-old classification problem, https://www.quantamagazine.org/mathematicians-solve-decades-old-classification-problem-20210805/.

Being able to tell if two mathematical structures are isomorphic is the basis for being able to classify them according to that structure. An abelian group is torsion-free if any finite sum of copies of a nonzero element is always nonzero. How can you tell if two countable torsion-free abelian groups are isomorphic? Well, Gianluca Paolini (U. of Turin) and Saharon Shelah (Hebrew University) have shown that it is "maximally hard" (Borel complete) to do that—just as hard as to decide if two infinite graphs are isomorphic.

Grohe, M., Schweitzer, P. (2020). The graph isomorphism problem, *Commun. Assoc. Comput. Mach.* 63(11): 128–134. https://cacm.acm.org/magazines/2020/11/248220-the-graph-isomorphism-problem/fulltext.

Determining whether two graphs are isomorphic is a problem in class NP that is not NP-complete (though there is an $\mathcal{O}(n \log n)$ algorithm for planar graphs). A recent result by László Babai (U. of Chicago) is an algorithm for graphs with n vertices that runs in *quasipolynomial time* $n^{p(\log n)}$ for some polynomial p. This article summarizes the history of the problem and previous results.

Schmitz, M. (2021). A plea for finite calculus. *Coll. Math. J.* 52 (2): 94–105.

Author Schmitz describes finite (discrete) calculus, which is based on summations and the difference operator, and elaborates some of its theory. He discusses its educational value as a subject in its own right and suggests that it could play an important pedagogical role in leading into infinitesimal calculus.

Chrisomalis, Stephen, Sequoyah and the almost-forgotten history of Cherokee numerals: The story of a numerical system nearly consigned to oblivion, adapted from *Reckonings: Numerals, Cognition, and History* (MIT Press, 2021). https://thereader.mitpress.mit.edu/sequoyah-and-the-almost-forgotten-history-of-cherokee-numerals/.

Sequoyah (c. 1770–1843) is famous for devising a script for the Cherokee language (each symbol corresponds not to a phoneme but to a syllable). He later proposed a set of numerals that did not become well-known. The system was not based on place value (units, tens, hundreds, . . .), despite the fact that verbal numerals in Cherokee are an ordinary decimal system. Instead, in his system "there are separate signs for each decade and unit, which combine together, so that 67 would be the sign for 60 followed by 7, rather than 6 followed by 7 as in Western numerals." Author Chrisomalis speculates that "there are many numerical systems for which evidence does not survive—or has yet to be found."